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# KINEMATICAL LIE ALGEBRAS IN $2 + 1$ DIMENSIONS

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**ABSTRACT.** We classify kinematical Lie algebras in dimension  $2 + 1$ . This is approached via the classification of deformations of the static kinematical Lie algebra. In addition, we determine which kinematical Lie algebras admit invariant symmetric inner products.

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## 1. INTRODUCTION

One consequence of the principle of relativity, which from a purely mathematical standpoint can be considered an instance of Klein's Erlanger Programme, is that the geometry of the universe is dictated by its Lie group of automorphisms. As in Klein's programme, by geometry one does not necessarily mean a metric geometry, but any sort of geometrical datum which the automorphisms leave invariant. In the context of relativity, for example, the Newtonian model of the universe, as an affine bundle (with three-dimensional fibres) over an affine line, has the galilean group as automorphisms and the invariant notions are time intervals between events and the euclidean distance between simultaneous events. By contrast, Minkowski spacetime has the Poincaré group as the group of automorphisms and the invariant notion is the proper distance (or, equivalently, the proper time). Both the galilean and Poincaré groups are examples of kinematical Lie groups, whose Lie algebras (in dimension  $2 + 1$ ) are the subject of this paper.

By a **kinematical Lie algebra** in dimension  $D$ , we mean a real  $\frac{1}{2}(D+1)(D+2)$ -dimensional Lie algebra with generators  $R_{ab} = -R_{ba}$ , with  $1 \leq a, b \leq D$ , spanning a Lie subalgebra isomorphic to  $\mathfrak{so}(D)$ :

$$[R_{ab}, R_{cd}] = \delta_{bc}R_{ad} - \delta_{ac}R_{bd} - \delta_{bd}R_{ac} + \delta_{ad}R_{bc}, \quad (1)$$

and  $B_a$ ,  $P_a$  and  $H$  which transform according to the vector, vector and scalar representations of  $\mathfrak{so}(D)$ , respectively – namely,

$$\begin{aligned} [R_{ab}, B_c] &= \delta_{bc} B_a - \delta_{ac} B_b \\ [R_{ab}, P_c] &= \delta_{bc} P_a - \delta_{ac} P_b \\ [R_{ab}, H] &= 0. \end{aligned} \tag{2}$$

The rest of the brackets between  $B_a$ ,  $P_a$  and  $H$  are only subject to the Jacobi identity: in particular, they must be  $\mathfrak{so}(D)$ -equivariant. The kinematical Lie algebra where those additional Lie brackets vanish is called the **static** kinematical Lie algebra, of which, by definition, every other kinematical Lie algebra is a deformation.

Up to isomorphism, there is only one kinematical Lie algebra in  $D = 0$ : it is one-dimensional and hence abelian. For  $D = 1$ , there are no rotations and hence any three-dimensional Lie algebra is kinematical. The classification is therefore the same as the celebrated Bianchi classification of three-dimensional real Lie algebras [1]. The classification for  $D = 3$  is due to Bacry and Nuyts [2] who completed earlier work of Bacry and Lévy-Leblond [3]. A deformation theory approach to the classification is described in [4], which completes earlier work [5] for the galilean and Bargmann algebras, and which also contains the classification of deformations of the universal central extension of the static kinematical Lie algebra. This approach is used in [6] to classify the kinematical Lie algebras for  $D > 3$  with and without central extension. The purpose of this paper is to solve the classification problem for  $D = 2$ . This problem is technically more involved than the problem for higher  $D$  for the simple reason that the representation of  $\mathfrak{so}(D)$  on  $\mathbb{R}^D$  has a larger endomorphism ring for  $D = 2$  than it does for any  $D > 2$ . Indeed, despite being a real irreducible representation, its endomorphism ring is the complex numbers. This means that it is often convenient to work not with real Lie algebras as for  $D \geq 3$ , but with *complexifications* of real Lie algebras; that is, complex Lie algebras with real structures. In order to trace a path of least effort, we will freely move from one description to another in this paper. Sufficient information is given to allow the reader to translate to their favourite formalism.

Let us remark in passing that the universal central extension of the static kinematical Lie algebra in  $D = 2$  is also larger than in  $D \geq 3$ . Whereas in  $D \geq 3$  there is a one-dimensional central subspace, in  $D = 2$  there is a five-dimensional central subspace spanned by  $Z_1, \dots, Z_5$  and brackets:

$$[B_a, B_b] = \epsilon_{ab} Z_1 \quad [B_a, P_b] = \delta_{ab} Z_2 + \epsilon_{ab} Z_3 \quad [P_a, P_b] = \epsilon_{ab} Z_4 \quad \text{and} \quad [R_{ab}, H] = \epsilon_{ab} Z_5. \tag{3}$$

The deformation problem of this centrally extended Lie algebra, while potentially interesting, is beyond the scope of this paper.

We refer to [4] for details on the methodology and for a brief review of the basic notions of deformation theory and Lie algebra cohomology, following [7], [8] and [9]. In this approach, we describe a Lie algebra structure on a vector space  $V$  as an element  $\mu_0 \in \Lambda^2 V^* \otimes V$  which has vanishing Nijenhuis–Richardson bracket with itself  $[\mu_0, \mu_0]$ . This bracket gives  $L^\bullet := \Lambda^{\bullet+1} V^* \otimes V$  the structure of a graded Lie superalgebra. In particular, the component  $[-, -] : L^1 \times L^1 \rightarrow L^2$  of the bracket is symmetric. Any other Lie algebra structure  $g = (V, \mu)$  defines  $\varphi = \mu - \mu_0 \in L^1$  which satisfies the Maurer–Cartan equation:

$$\partial\varphi = \frac{1}{2} [\varphi, \varphi], \tag{4}$$

where  $\partial\varphi := -[\mu_0, \varphi]$  is one component of the Chevalley–Eilenberg differential of the Lie algebra  $g_0 = (V, \mu_0)$  with values in the adjoint representation. The deformation theory approach is to solve the Maurer–Cartan equation perturbatively, by writing  $\varphi$  as a formal power series  $\varphi = \sum_{n=1}^{\infty} t^n \varphi_n$  and solving equation (4) order by order in  $t$ . The first order equation says that  $\partial\varphi_1 = 0$ . We call such cocycles **infinitesimal deformations** and each such  $\varphi_1$  defines a class in  $H^2(g_0; g_0)$ . If this class is zero, then  $\varphi_1$  is tangent to the  $GL(V)$  orbit of  $\mu_0$  and we say that the infinitesimal deformation is ineffective. Therefore the interesting infinitesimal deformations are those which are not cohomologically trivial. In practice we parametrise the space of infinitesimal deformations by splitting the cohomology sequence  $B^2 \rightarrow Z^2 \rightarrow H^2(g_0; g_0)$  and choosing a convenient complement  $\mathcal{H}^2 \subset Z^2$  to the coboundaries  $B^2$  in the space  $Z^2$  of cocycles. The higher order terms in the Maurer–Cartan equation (4) can be understood as a sequence of obstructions to integrating the infinitesimal deformation  $\varphi_1$ . At every order in the perturbation expansion of the Maurer–Cartan equation we find a cohomology class in  $H^3$ , whose vanishing is a condition *sine qua non* to be able to continue integrating the deformation. Although this process could in principle continue indefinitely, it seldom does and indeed the deformations in this paper are either obstructed or integrable at second order in the perturbative expansion.

In this paper we are interested only in deformations of the static kinematical Lie algebra which are themselves kinematical: i.e., such that the Lie brackets  $[R, -]$  involving the rotational generator  $R := -\frac{1}{2} \epsilon_{ab} R_{ab}$  are not modified or, equivalently, that the deformation  $\varphi$  obeys  $\varphi(R, -) = 0$ . In other words,

if we let  $\mathfrak{r}$  denote the Lie subalgebra of the static kinematical Lie algebra spanned by  $R$  and  $\mathfrak{h}$  the complementary ideal spanned by  $B_a, P_a$  and  $H$ , then<sup>1</sup>  $\varphi \in \Lambda^2 \mathfrak{h}^* \otimes \mathfrak{g}$ . In other words, the relevant deformation complex is the relative subcomplex  $C^\bullet(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$  which consists of the  $\mathfrak{r}$ -invariant cochains in  $\Lambda^\bullet \mathfrak{h}^* \otimes \mathfrak{g}$ . For  $D \geq 3$ , the relative subcomplex is quasi-isomorphic to the full deformation complex, as a consequence of the Hochschild–Serre decomposition theorem [9], and therefore all deformations of the static kinematical Lie algebra are automatically kinematical themselves. This theorem is not applicable for  $D = 2$  because  $\mathfrak{r}$  is not semisimple here. As a result there are in principle deformations which are not kinematical. (In fact, the space of all infinitesimal deformations is 19-dimensional, whereas as we will see the space corresponding to kinematical deformations is “only” 11-dimensional.)

An important characteristic of a Lie algebra, particularly for applications in field theory, is whether or not the Lie algebra admits a symmetric inner product which is invariant under the adjoint action of the Lie algebra on itself. Such Lie algebras are said to be *metric*. In this paper we also determine which kinematical Lie algebras are metric. We will see that similar to what happens in  $D = 3$  and contrary to what happens in dimension  $D > 3$ , there are non-semisimple metric kinematical Lie algebras.

The plan of this paper is the following. In Section 2 we describe the deformation complex, but we relegate to Appendix A the precise enumeration of cochains that we will use in our calculations, as well as the relevant component of the Nijenhuis–Richardson bracket. There are two complementary descriptions of kinematical Lie algebras in this dimension: one is as real Lie algebras and the other as complex Lie algebras with a real structure. This second description simplifies the discussion of automorphisms, which will play a crucial rôle in this approach. In Section 3 we describe the group  $G$  of automorphisms of  $\mathfrak{g}$  which preserve the deformation complex. This will play an important rôle when we split the cohomology sequence to parametrise the space of infinitesimal deformations, when we solve the obstruction relations and also when we classify the different integrable deformations up to isomorphism. In Section 4 we calculate the second cohomology of the deformation complex and choose a convenient  $G$ -stable parametrisation of the infinitesimal deformations, whose obstructions are analysed in Section 5. We find that integrable deformations are of at most second order and they fall into one of four branches labelling the  $G$ -orbits in a four-dimensional subspace of the cohomology. In Section 6 we study the isomorphism classes of integrable deformations for each of those branches. The main technique is to exploit the stabiliser of the typical point in each orbit to bring the remaining free parameters to a canonical form. Doing so for each orbit we arrive at the classification which is summarised in Table 1 in Section 8, which also contains the information of which deformations are metric, as determined in Section 7.

## 2. THE DEFORMATION COMPLEX

Let  $\mathfrak{g}$  be the static kinematical Lie algebra for  $D = 2$ . It is spanned by  $R, B_a, P_a, H$  subject to the following nonzero Lie brackets:

$$[R, B_a] = \epsilon_{ab} B_b \quad \text{and} \quad [R, P_a] = \epsilon_{ab} P_b. \quad (5)$$

Let  $\mathfrak{r} \cong \mathfrak{so}(2)$  denote the abelian Lie subalgebra spanned by  $R$  and let  $\mathfrak{h}$  denote the abelian ideal spanned by  $B_a, P_a, H$ . Let  $\beta_a, \pi_a, \eta$  denote the canonical dual basis for  $\mathfrak{h}^*$ .

We may diagonalise the action of  $R$  by complexifying  $\mathfrak{g}$ . This will turn out to simplify the action of automorphisms on the deformation complex, so we will also describe this approach. To this end we introduce  $\mathbf{B} = B_1 + iB_2$  and  $\mathbf{P} = P_1 + iP_2$  and extend the Lie brackets complex-linearly, so that now

$$[R, \mathbf{B}] = -i\mathbf{B} \quad \text{and} \quad [R, \mathbf{P}] = -i\mathbf{P}. \quad (6)$$

We also have  $\bar{\mathbf{B}} = B_1 - iB_2$  and  $\bar{\mathbf{P}} = P_1 - iP_2$ , which satisfy

$$[R, \bar{\mathbf{B}}] = i\bar{\mathbf{B}} \quad \text{and} \quad [R, \bar{\mathbf{P}}] = i\bar{\mathbf{P}}. \quad (7)$$

The complex span of  $R, H, \mathbf{B}, \mathbf{P}, \bar{\mathbf{B}}, \bar{\mathbf{P}}$ , which we denote by  $\mathbb{C} \langle R, H, \mathbf{B}, \mathbf{P}, \bar{\mathbf{B}}, \bar{\mathbf{P}} \rangle$ , defines a complex Lie algebra  $\mathfrak{g}_{\mathbb{C}}$ . This complex Lie algebra has a conjugation (that is, a complex-antilinear involutive automorphism) denoted by  $\star$  and defined by  $H^\star = H, R^\star = R, \mathbf{B}^\star = \bar{\mathbf{B}}$  and  $\mathbf{P}^\star = \bar{\mathbf{P}}$ . We see that the real Lie subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  consisting of real elements (i.e., those  $X \in \mathfrak{g}_{\mathbb{C}}$  such that  $X^\star = X$ ) is the static kinematical Lie algebra  $\mathfrak{g}$ .

Let  $\mathfrak{h}_{\mathbb{C}}$  denote the ideal of  $\mathfrak{g}_{\mathbb{C}}$  spanned by  $H, \mathbf{B}, \mathbf{P}, \bar{\mathbf{B}}, \bar{\mathbf{P}}$ . We will let  $\eta, \beta, \pi, \bar{\beta}, \bar{\pi}$  denote the canonical dual basis for  $\mathfrak{h}_{\mathbb{C}}^*$ . These are related to  $\beta_a$  and  $\pi_a$  by the following relations:

$$\beta = \frac{1}{2}(\beta_1 - i\beta_2) \quad \text{and} \quad \pi = \frac{1}{2}(\pi_1 - i\pi_2), \quad (8)$$

with  $\bar{\beta}$  and  $\bar{\pi}$  being their naive complex conjugates. We extend the action of the conjugation  $\star$  to  $\mathfrak{h}_{\mathbb{C}}^*$  by  $\eta^\star = \eta, \beta^\star = \bar{\beta}$  and  $\pi^\star = \bar{\pi}$ .

<sup>1</sup>We will denote the static Lie algebra by  $\mathfrak{g}$ , rather than  $\mathfrak{g}_0$ , in an effort not to overburden ourselves notationally.

We are interested in kinematical Lie algebras, so we are not deforming the Lie brackets involving the rotation generator; that is,  $B_a$  and  $P_a$  still transform as vectors and  $H$  still transforms as a scalar. The Jacobi identity then says that the Lie brackets must be  $\mathfrak{r}$ -equivariant. This implies that the deformation complex is thus  $C^\bullet := C^\bullet(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}) \cong (\wedge^\bullet \mathfrak{h}^* \otimes \mathfrak{g})^\mathfrak{r}$ , which can also be identified with the  $\mathfrak{r}$ -invariant subcomplex of the Chevalley–Eilenberg complex of the abelian Lie algebra  $\mathfrak{h}$  with values in the representation  $\mathfrak{g}$ .

The deformation complex  $C^\bullet$  can also be identified with the real subcomplex of  $C_\mathbb{C}^\bullet := (\wedge^\bullet \mathfrak{h}_\mathbb{C}^* \otimes \mathfrak{g}_\mathbb{C})^\mathfrak{r}$ . This real subcomplex consists of those cochains which are fixed by the conjugation  $\star$ . At a practical level, one can work with  $C^\bullet$  by working with  $C_\mathbb{C}^\bullet$  and making sure that one considers only real elements. This turns out to be very convenient when discussing automorphisms, since these act more simply and more naturally on  $C_\mathbb{C}^\bullet$ .

The real dimension of  $C^\bullet$  is the complex dimension of  $C_\mathbb{C}^\bullet$ , which can be calculated using character theory as follows. The character  $\chi_{\mathfrak{g}_\mathbb{C}}(q)$  of  $\mathfrak{g}_\mathbb{C}$  as a representation of  $\mathfrak{r}$  is given by

$$\chi_{\mathfrak{g}_\mathbb{C}}(q) = 2 + 2(q + q^{-1}), \quad (9)$$

whereas that of  $\mathfrak{h}_\mathbb{C}$  is given by

$$\chi_{\mathfrak{h}_\mathbb{C}}(q) = 1 + 2(q + q^{-1}). \quad (10)$$

Since this is invariant under  $q \mapsto q^{-1}$ , this is also the character  $\chi_{\mathfrak{h}_\mathbb{C}^*}$  of  $\mathfrak{h}_\mathbb{C}^*$ . The character of  $\wedge^p \mathfrak{h}_\mathbb{C}^* \otimes \mathfrak{g}_\mathbb{C}$  can be calculated as follows. First of all, since characters are multiplicative over the tensor product,

$$\chi_{\wedge^p \mathfrak{h}_\mathbb{C}^* \otimes \mathfrak{g}_\mathbb{C}}(q) = \chi_{\wedge^p \mathfrak{h}_\mathbb{C}^*}(q) \chi_{\mathfrak{g}_\mathbb{C}}(q) = 2(1 + q + q^{-1}) \chi_{\wedge^p \mathfrak{h}_\mathbb{C}}(q). \quad (11)$$

The character for the  $\wedge^p \mathfrak{h}_\mathbb{C}$  can be read off from their generating function:

$$\sum_{n=0}^{\infty} t^n \chi_{\wedge^n \mathfrak{h}_\mathbb{C}}(q) = \exp \left( - \sum_{\ell=1}^{\infty} \frac{(-t)^\ell}{\ell} \chi_{\mathfrak{h}_\mathbb{C}}(q^\ell) \right). \quad (12)$$

Expanding this to second order we find that  $\chi_{\wedge^0 \mathfrak{h}_\mathbb{C}}(q) = 1$ ,  $\chi_{\wedge^1 \mathfrak{h}_\mathbb{C}}(q) = \chi_{\mathfrak{h}_\mathbb{C}}(q)$  and

$$\chi_{\wedge^2 \mathfrak{h}_\mathbb{C}}(q) = \frac{1}{2} (\chi_{\mathfrak{h}_\mathbb{C}}(q)^2 - \chi_{\mathfrak{h}_\mathbb{C}}(q^2)) = q^{-2} + 2q^{-1} + 4 + 2q + q^2, \quad (13)$$

and by Poincaré duality  $\chi_{\wedge^3 \mathfrak{h}_\mathbb{C}}(q) = \chi_{\wedge^2 \mathfrak{h}_\mathbb{C}}(q)$ . Therefore,

$$\begin{aligned} \chi_{\wedge^0 \mathfrak{h}_\mathbb{C}^* \otimes \mathfrak{g}_\mathbb{C}}(q) &= 2q^{-1} + 2 + 2q \implies \dim C^0 = 2 \\ \chi_{\wedge^1 \mathfrak{h}_\mathbb{C}^* \otimes \mathfrak{g}_\mathbb{C}}(q) &= 4q^{-2} + 6q^{-1} + 10 + 6q + 4q^2 \implies \dim C^1 = 10 \\ \chi_{\wedge^2 \mathfrak{h}_\mathbb{C}^* \otimes \mathfrak{g}_\mathbb{C}}(q) &= 2q^{-3} + 6q^{-2} + 14q^{-1} + 16 + 14q + 6q^2 + 2q^3 \implies \dim C^2 = 16, \end{aligned} \quad (14)$$

and again  $\dim C^3 = 16$  by duality.

In Appendix A we define bases for the  $C^0, \dots, C^3$  and  $C_\mathbb{C}^0, \dots, C_\mathbb{C}^3$ , as well as a dictionary between the two bases. We also tabulate the Nijenhuis–Richardson product on the space of 2-cochains, which will be useful when computing the obstructions to infinitesimal deformations.

The Chevalley–Eilenberg differential on  $C^\bullet$  is defined on generators by

$$\partial R = -\epsilon_{ab}(\beta_a B_b + \pi_a P_b) \quad \partial \beta_a = \partial \pi_a = \partial \eta = \partial B_a = \partial P_a = \partial H = 0, \quad (15)$$

and the one on  $C_\mathbb{C}^\bullet$  is given by

$$\partial R = i\beta B - i\bar{\beta} \bar{B} + i\pi P - i\bar{\pi} \bar{P} \quad \text{and} \quad \partial \beta = \partial \pi = \partial \eta = \partial B = \partial P = \partial H = 0. \quad (16)$$

The differential is real, so that  $\partial \bar{B} = \overline{\partial B} = 0$ , et cetera, so the real elements of  $C_\mathbb{C}^\bullet$  do indeed form a subcomplex. From the above formulae it is easy to calculate the differential on the bases given in Appendix A.

### 3. AUTOMORPHISMS

For the static kinematical Lie algebra in dimension  $D \geq 3$ , the subgroup of automorphisms of  $\mathfrak{g}$  which preserves the deformation complex is  $GL(\mathbb{R}^2) \times \mathbb{R}^\times$  (see, e.g., [4, 6]). For  $D = 2$  this is enhanced to  $GL(\mathbb{C}^2) \times \mathbb{R}^\times$ . This is transparent in the complex version of the Lie algebra, where the action of  $G$  on the generators of  $\mathfrak{g}_\mathbb{C}$  is given by declaring  $R$  to be invariant and by

$$(\mathbf{B}, \mathbf{P}, H) \mapsto (\mathbf{B}, \mathbf{P}, H) \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{C}^2) \quad \text{and} \quad \lambda \in \mathbb{R}^\times, \quad (17)$$

with the induced action on the generators of  $\mathfrak{h}_\mathbb{C}^*$  is given by the transpose inverse:

$$(\beta, \pi, \eta) \mapsto (\beta, \pi, \eta) \begin{pmatrix} d/\Delta & -c/\Delta & 0 \\ -b/\Delta & a/\Delta & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \quad \text{where} \quad \Delta = ad - bc. \quad (18)$$

In order to ensure that the automorphisms preserve the real deformation complex, we must define the action of  $G$  on  $\mathbf{B}, \mathbf{P}, \beta, \pi$  simply by complex conjugating the above formulae.

In summary, and being more explicit,

$$\begin{aligned} \mathbf{B} &\mapsto \mathbf{a}\mathbf{B} + \mathbf{c}\mathbf{P} & \beta &\mapsto \Delta^{-1}(\mathbf{d}\beta - \mathbf{b}\pi) \\ \mathbf{P} &\mapsto \mathbf{b}\mathbf{B} + \mathbf{d}\mathbf{P} & \pi &\mapsto \Delta^{-1}(-\mathbf{c}\beta + \mathbf{a}\pi), \\ \mathbf{H} &\mapsto \lambda\mathbf{H} & \eta &\mapsto \lambda^{-1}\eta \end{aligned} \quad (19)$$

with  $\mathbf{R}$  invariant and  $\bar{\mathbf{B}} \mapsto \bar{\mathbf{a}}\bar{\mathbf{B}} + \bar{\mathbf{c}}\bar{\mathbf{P}}$ , et cetera.

From this we can work out the action of  $G$  on the bases given in Appendix A.2. For  $C_{\mathbb{C}}^1$  we find

$$\begin{aligned} \mathbf{a}_1 &\mapsto \lambda^{-1}\mathbf{a}_1 & \mathbf{a}_3 - \mathbf{a}_6 &\mapsto \Delta^{-1}((\mathbf{a}\mathbf{d} + \mathbf{b}\mathbf{c})(\mathbf{a}_3 - \mathbf{a}_6) + 2\mathbf{c}\mathbf{d}\mathbf{a}_4 - 2\mathbf{a}\mathbf{b}\mathbf{a}_5) \\ \mathbf{a}_2 &\mapsto \mathbf{a}_2 & \mathbf{a}_4 &\mapsto \Delta^{-1}(\mathbf{b}\mathbf{d}(\mathbf{a}_3 - \mathbf{a}_6) + \mathbf{d}^2\mathbf{a}_4 - \mathbf{b}^2\mathbf{a}_5) \\ \mathbf{a}_3 + \mathbf{a}_6 &\mapsto \mathbf{a}_3 + \mathbf{a}_6 & \mathbf{a}_5 &\mapsto \Delta^{-1}(-\mathbf{a}\mathbf{c}(\mathbf{a}_3 - \mathbf{a}_6) - \mathbf{c}^2\mathbf{a}_4 + \mathbf{a}^2\mathbf{a}_5) \end{aligned} \quad (20)$$

and for  $C_{\mathbb{C}}^2$  we find

$$\begin{aligned} \mathbf{c}_1 + \mathbf{c}_4 &\mapsto \lambda^{-1}(\mathbf{c}_1 + \mathbf{c}_4) \\ \mathbf{c}_1 - \mathbf{c}_4 &\mapsto \lambda^{-1}\Delta^{-1}((\mathbf{a}\mathbf{d} + \mathbf{b}\mathbf{c})(\mathbf{c}_1 - \mathbf{c}_4) + 2\mathbf{c}\mathbf{d}\mathbf{c}_2 - 2\mathbf{a}\mathbf{b}\mathbf{c}_3) \\ \mathbf{c}_2 &\mapsto \lambda^{-1}\Delta^{-1}(\mathbf{b}\mathbf{d}(\mathbf{c}_1 - \mathbf{c}_4) + \mathbf{d}^2\mathbf{c}_2 - \mathbf{b}^2\mathbf{c}_3) \\ \mathbf{c}_3 &\mapsto \lambda^{-1}\Delta^{-1}(-\mathbf{a}\mathbf{c}(\mathbf{c}_1 - \mathbf{c}_4) - \mathbf{c}^2\mathbf{c}_2 + \mathbf{a}^2\mathbf{c}_3) \\ \mathbf{c}_5 &\mapsto |\Delta|^{-2}(|\mathbf{d}|^2\mathbf{c}_5 - \mathbf{i}\bar{\mathbf{b}}\mathbf{d}\mathbf{c}_7 + \mathbf{i}\mathbf{b}\bar{\mathbf{d}}\mathbf{c}_7 + |\mathbf{b}|^2\mathbf{c}_9) \\ \mathbf{c}_7 &\mapsto |\Delta|^{-2}(\mathbf{i}\bar{\mathbf{c}}\mathbf{d}\mathbf{c}_5 + \bar{\mathbf{a}}\mathbf{d}\mathbf{c}_7 - \mathbf{b}\bar{\mathbf{c}}\mathbf{c}_7 + \mathbf{i}\bar{\mathbf{a}}\mathbf{b}\mathbf{c}_9) \\ \mathbf{c}_9 &\mapsto |\Delta|^{-2}(|\mathbf{c}|^2\mathbf{c}_5 - \mathbf{i}\bar{\mathbf{a}}\mathbf{c}\mathbf{c}_7 + \mathbf{i}\mathbf{a}\bar{\mathbf{c}}\mathbf{c}_7 + |\mathbf{a}|^2\mathbf{c}_9) \\ \mathbf{c}_6 &\mapsto \lambda|\Delta|^{-2}(|\mathbf{d}|^2\mathbf{c}_6 - \mathbf{i}\bar{\mathbf{b}}\mathbf{d}\mathbf{c}_8 + \mathbf{i}\mathbf{b}\bar{\mathbf{d}}\mathbf{c}_8 + |\mathbf{b}|^2\mathbf{c}_{10}) \\ \mathbf{c}_8 &\mapsto \lambda|\Delta|^{-2}(\mathbf{i}\bar{\mathbf{c}}\mathbf{d}\mathbf{c}_6 + \bar{\mathbf{a}}\mathbf{d}\mathbf{c}_8 - \mathbf{b}\bar{\mathbf{c}}\mathbf{c}_8 + \mathbf{i}\bar{\mathbf{a}}\mathbf{b}\mathbf{c}_{10}) \\ \mathbf{c}_{10} &\mapsto \lambda|\Delta|^{-2}(|\mathbf{c}|^2\mathbf{c}_6 - \mathbf{i}\bar{\mathbf{a}}\mathbf{c}\mathbf{c}_8 + \mathbf{i}\mathbf{a}\bar{\mathbf{c}}\mathbf{c}_8 + |\mathbf{a}|^2\mathbf{c}_{10}). \end{aligned} \quad (21)$$

Let us point out that the representation of  $G$  on the four-dimensional complex vector space with ordered basis  $(\mathbf{c}_{10}, \mathbf{i}\bar{\mathbf{c}}_8, -\mathbf{i}\mathbf{c}_8, \mathbf{c}_6)$  is such that  $(A, \lambda) \in G$  acts via the matrix  $\lambda M_A$ , where

$$M_A := \frac{1}{|\Delta|^2} \begin{pmatrix} |\mathbf{a}|^2 & \mathbf{a}\bar{\mathbf{b}} & \mathbf{b}\bar{\mathbf{a}} & |\mathbf{b}|^2 \\ \mathbf{a}\bar{\mathbf{c}} & \mathbf{a}\bar{\mathbf{d}} & \mathbf{b}\bar{\mathbf{c}} & \mathbf{b}\bar{\mathbf{d}} \\ \mathbf{c}\bar{\mathbf{a}} & \mathbf{c}\bar{\mathbf{b}} & \mathbf{d}\bar{\mathbf{a}} & \mathbf{d}\bar{\mathbf{b}} \\ |\mathbf{c}|^2 & \mathbf{c}\bar{\mathbf{d}} & \mathbf{d}\bar{\mathbf{c}} & |\mathbf{d}|^2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix} \otimes \frac{1}{\bar{\Delta}} \begin{pmatrix} \bar{\mathbf{a}} & \bar{\mathbf{b}} \\ \bar{\mathbf{c}} & \bar{\mathbf{d}} \end{pmatrix}, \quad (22)$$

which shows that the representation of  $GL(\mathbb{C}^2)$  which sends  $A \mapsto M_A$  is isomorphic to  $(\Lambda^2 E^* \otimes E) \otimes (\Lambda^2 \bar{E}^* \otimes \bar{E})$ , where  $E = \mathbb{C}^2$  is the identity representation of  $GL(\mathbb{C}^2)$  and  $\bar{E}$  the conjugate representation. In the symmetric square of this representation there is a submodule isomorphic to  $(\Lambda^2 E^*)^2 \otimes \Lambda^2 E \otimes (\Lambda^2 \bar{E}^*)^2 \otimes \Lambda^2 \bar{E} \cong \Lambda^2 E^* \otimes \Lambda^2 \bar{E}^*$  and this means that there is a symmetric bilinear form  $K$  which obeys

$$M_A^T K M_A = |\Delta|^{-2} K. \quad (23)$$

Relative to the basis  $(\mathbf{c}_{10}, \mathbf{i}\bar{\mathbf{c}}_8, -\mathbf{i}\mathbf{c}_8, \mathbf{c}_6)$ , the matrix  $K$  is given (up to a scale) by

$$K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

The action on the real basis for  $C^1$  and  $C^2$  is more cumbersome and we will not write it down. Our strategy shall be that we will calculate infinitesimal deformations and obstructions using the real complex  $C^\bullet$  and the real basis, but shall complexify to  $C_{\mathbb{C}}^\bullet$  and use the complex basis when discussing the action of automorphisms.

From (21) it follows that  $C_{\mathbb{C}}^2$  decomposes into the following complex  $G$ -submodules:

$$C_{\mathbb{C}}^2 = \mathbb{C} \langle \mathbf{c}_1 + \mathbf{c}_4 \rangle \oplus \mathbb{C} \langle \mathbf{c}_1 - \mathbf{c}_4, \mathbf{c}_2, \mathbf{c}_3 \rangle \oplus \mathbb{C} \langle \bar{\mathbf{c}}_1 - \bar{\mathbf{c}}_4, \bar{\mathbf{c}}_2, \bar{\mathbf{c}}_3 \rangle \oplus \mathbb{C} \langle \mathbf{c}_5, \mathbf{c}_7, \bar{\mathbf{c}}_7, \mathbf{c}_9 \rangle \oplus \mathbb{C} \langle \mathbf{c}_6, \mathbf{c}_8, \bar{\mathbf{c}}_8, \mathbf{c}_{10} \rangle, \quad (25)$$

and, in turn, this decomposes the real subspace  $C^2$  into the following real  $G$ -submodules:

$$\begin{aligned} C^2 = \mathbb{R} \langle \text{Re}(\mathbf{c}_1 + \mathbf{c}_4) \rangle \oplus \mathbb{R} \langle \text{Im}(\mathbf{c}_1 + \mathbf{c}_4) \rangle \oplus \mathbb{R} \langle \text{Re}(\mathbf{c}_1 - \mathbf{c}_4), \text{Im}(\mathbf{c}_1 - \mathbf{c}_4), \text{Re} \mathbf{c}_2, \text{Im} \mathbf{c}_2, \text{Re} \mathbf{c}_3, \text{Im} \mathbf{c}_3 \rangle \\ \oplus \mathbb{R} \langle \mathbf{c}_5, \text{Re} \mathbf{c}_7, \text{Im} \mathbf{c}_7, \mathbf{c}_9 \rangle \oplus \mathbb{R} \langle \mathbf{c}_6, \text{Re} \mathbf{c}_8, \text{Im} \mathbf{c}_8, \mathbf{c}_{10} \rangle, \end{aligned} \quad (26)$$



or in terms of the real basis,

$$\begin{aligned} C^2 = \mathbb{R} \langle c_1 + c_7 \rangle \oplus \mathbb{R} \langle c_2 + c_8 \rangle \oplus \mathbb{R} \langle c_1 - c_7, c_2 - c_8, c_3, c_4, c_5, c_6 \rangle \\ \oplus \mathbb{R} \langle c_9, c_{11}, c_{12}, c_{15} \rangle \oplus \mathbb{R} \langle c_{10}, c_{13}, c_{14}, c_{16} \rangle. \end{aligned} \quad (27)$$

#### 4. INFINITESIMAL DEFORMATIONS

Infinitesimal (kinematical) deformations of the static Lie algebra  $\mathfrak{g}$  are classified by  $H^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ , which as explained above is isomorphic to  $H^2(\mathfrak{h}; \mathfrak{g})^\tau$ . From the expression of the Chevalley–Eilenberg differential on generators given in equation (15), we can compute the spaces of cocycles and coboundaries in low degree. Recall that  $C^0$  is spanned by  $R$  and  $H$ . Clearly  $H$  is a cocycle, so  $B^1$  is spanned by  $\partial R = -a_4 - a_{10} = -2 \operatorname{Im}(a_3 + a_6)$ . We see from equation (20) that  $\partial R$  is indeed invariant under  $G$ . The differential  $\partial : C^1 \rightarrow C^2$  is given by  $\partial a_1 = c_2 + c_8 = 2 \operatorname{Im}(c_1 + c_4)$  and zero on the other basis cochains. Therefore  $B^2$  is spanned by  $\operatorname{Im}(c_1 + c_4)$ , which from (21) we see that it is a  $G$ -submodule, as expected. The differential  $\partial : C^2 \rightarrow C^3$  is given by

$$\partial c_9 = b_{14}, \quad \partial c_{11} = \frac{1}{2}(b_{12} - b_9), \quad \partial c_{12} = b_{16} - b_{13} \quad \text{and} \quad \partial c_{15} = -b_{15}, \quad (28)$$

and zero on the other basis cochains. Therefore,

$$Z^2 = \mathbb{R} \langle c_1, \dots, c_8, c_{10}, c_{13}, c_{14}, c_{16} \rangle. \quad (29)$$

We wish to split the sequence

$$0 \longrightarrow B^2 \longrightarrow Z^2 \longrightarrow H^2 \longrightarrow 0 \quad (30)$$

by choosing a subspace  $\mathcal{H}^2 \subset Z^2$  which is stable under the action of the group  $G$  of automorphisms. From the explicit decomposition of  $C^2$  as  $G$ -submodules in (26), we find that the subspace  $\mathcal{H}^2$  can be chosen to be the following direct sum of  $G$ -submodules of  $Z^2$ :

$$\begin{aligned} \mathcal{H}^2 = \mathbb{R} \langle \operatorname{Re}(c_1 + c_4) \rangle \oplus \mathbb{R} \langle \operatorname{Re}(c_1 - c_4), \operatorname{Im}(c_1 - c_4), \operatorname{Re} c_2, \operatorname{Im} c_2, \operatorname{Re} c_3, \operatorname{Im} c_3 \rangle \\ \oplus \mathbb{R} \langle c_6, \operatorname{Re} c_8, \operatorname{Im} c_8, c_{10} \rangle, \end{aligned} \quad (31)$$

or in terms of the real basis

$$\mathcal{H}^2 = \mathbb{R} \langle c_1 + c_7 \rangle \oplus \mathbb{R} \langle c_1 - c_7, c_2 - c_8, c_3, c_4, c_5, c_6 \rangle \oplus \mathbb{R} \langle c_{10}, c_{13}, c_{14}, c_{16} \rangle. \quad (32)$$

We therefore have an 11-dimensional space of infinitesimal deformations, parametrised as:

$$p_1 = t_1(c_1 + c_7) + t_2(c_1 - c_7) + t_3 c_3 + t_4 c_5 + t_5(c_2 - c_8) + t_6 c_4 + t_7 c_6 + t_8 c_{14} + t_9 c_{10} + t_{10} c_{16} + t_{11} c_{13}, \quad (33)$$

where the order has been chosen for later computational convenience.

We claim that the action of  $G$  on the four-dimensional space of infinitesimal deformations parametrised by  $t_8, t_9, t_{10}, t_{11}$  is essentially a four-dimensional Lorentz transformation and a dilation. To see this, notice that this component of the deformation is given by

$$t_8 c_{14} + t_9 c_{10} + t_{10} c_{16} + t_{11} c_{13} = 2(-t_{10} c_{10} + (t_8 - it_{11})(-i\bar{c}_8) + (t_8 + it_{11})(-ic_8) - t_9 c_6), \quad (34)$$

using the dictionary in equation (116). As shown in Section 3, the action of  $G$  preserves the conformal class of the inner product defined by  $K$  in equation (24). The norm of  $t_8 c_{14} + t_9 c_{10} + t_{10} c_{16} + t_{11} c_{13}$  relative to that inner product is (up to an inconsequential factor):

$$(-t_{10}, t_8 - it_{11}, t_8 + it_{11}, -t_9)K(-t_{10}, t_8 - it_{11}, t_8 + it_{11}, -t_9)^T = t_8^2 - t_9 t_{10} + t_{11}^2, \quad (35)$$

which has lorentzian signature. There are four  $G$ -orbits in that four-dimensional vector space, labelled by the following choices for the vector  $\mathbf{t} = (t_8, t_9, t_{10}, t_{11})$ :

- (1) the **zero orbit** of the vector  $\mathbf{t} = (0, 0, 0, 0)$ ;
- (2) the **lightlike orbit** of the vector  $\mathbf{t} = (0, 1, 0, 0)$ ;
- (3) the **timelike orbit** of the vector  $\mathbf{t} = (0, 1, 1, 0)$ ; and
- (4) the **spacelike orbit** of the vector  $\mathbf{t} = (1, 0, 0, 0)$ .

Let us now consider the obstructions to integrating the infinitesimal deformations found above.

## 5. OBSTRUCTIONS

The first obstruction is the class of  $\frac{1}{2}[\varphi_1, \varphi_1]$  in  $H^3$ , which can be calculated from the explicit expression (113) for the Nijenhuis–Richardson bracket. Its vanishing in cohomology is equivalent to the following system of quadrics (after some simplification):

$$\begin{aligned} 0 &= 2t_5t_8 - t_7t_9 + t_6t_{10} & 0 &= t_1t_8 \\ 0 &= 2t_5t_{11} + t_4t_9 + t_3t_{10} & 0 &= t_1t_9 \\ 0 &= t_3t_8 + t_2t_9 - t_6t_{11} & 0 &= t_1t_{10} \\ 0 &= t_4t_8 - t_2t_{10} + t_7t_{11} & 0 &= t_1t_{11} \end{aligned} \quad (36)$$

Assuming these equations are satisfied,  $\frac{1}{2}[\varphi_1, \varphi_1] = \partial\varphi_2$ , where

$$\varphi_2 = (t_3t_{11} + t_5t_9 + t_6t_8)c_9 + (t_2t_8 - t_4t_9 - t_5t_{11})c_{11} + (-t_5t_8 - t_2t_{11} + t_7t_9)c_{12} + (t_7t_8 - t_5t_{10} - t_4t_{11})c_{15}. \quad (37)$$

The next obstruction is the class of  $[\varphi_1, \varphi_2]$  in  $H^3$ , which again can be calculated from (113). Demanding that this vanishes, we obtain a number of cubic equations, which together with the quadrics leads to some simplification:

$$\begin{aligned} 0 &= t_8(2t_2t_5 + t_4t_6 + t_3t_7) \\ 0 &= t_9(2t_2t_5 + t_4t_6 + t_3t_7) \\ 0 &= t_{10}(2t_2t_5 + t_4t_6 + t_3t_7) \\ 0 &= t_{11}(2t_2t_5 + t_4t_6 + t_3t_7); \end{aligned} \quad (38)$$

although only three are independent once the quadrics are taken into account. If these cubic equations are satisfied, it is not just the cohomology class of  $[\varphi_1, \varphi_2]$  which vanishes, but the cocycle itself. Therefore we can take  $\varphi_3 = 0$ . Finally, we see from (113) that the cochains  $c_9, c_{11}, c_{12}, c_{15}$  appearing in  $\varphi_2$  have vanishing Nijenhuis–Richardson brackets among themselves, so that also  $[\varphi_2, \varphi_2] = 0$  and hence the deformation integrates at second order.

In summary, we have the following deformation

$$\begin{aligned} \varphi &= t_1(c_1 + c_7) + t_2(c_1 - c_7) + t_3c_3 + t_4c_5 + t_5(c_2 - c_8) + t_6c_4 + t_7c_6 \\ &\quad + t_8c_{14} + t_9c_{10} + t_{10}c_{16} + t_{11}c_{13} + (t_3t_{11} + t_5t_9 + t_6t_8)c_9 + (t_2t_8 - t_4t_9 - t_5t_{11})c_{11} \\ &\quad + (t_7t_9 - t_2t_{11} - t_5t_8)c_{12} + (t_7t_8 - t_5t_{10} - t_4t_{11})c_{15} \end{aligned} \quad (39)$$

subject to the following integrability equations:

$$\begin{aligned} 0 &= t_1t_8 & 0 &= 2t_5t_8 - t_7t_9 + t_6t_{10} & 0 &= t_8(2t_2t_5 + t_4t_6 + t_3t_7) \\ 0 &= t_1t_9 & 0 &= 2t_5t_{11} + t_4t_9 + t_3t_{10} & 0 &= t_9(2t_2t_5 + t_4t_6 + t_3t_7) \\ 0 &= t_1t_{10} & 0 &= t_3t_8 + t_2t_9 - t_6t_{11} & 0 &= t_{10}(2t_2t_5 + t_4t_6 + t_3t_7) \\ 0 &= t_1t_{11} & 0 &= t_4t_8 - t_2t_{10} + t_7t_{11} & 0 &= t_{11}(2t_2t_5 + t_4t_6 + t_3t_7). \end{aligned} \quad (40)$$

## 6. DEFORMATIONS

While it is possible to solve the obstruction relations (40) using Gröbner methods, it is much more transparent to exploit the automorphisms and in particular the orbit decomposition discussed at the end of Section 4. This leads us to consider the four branches of solutions into which this section is divided.

**6.1. Zero orbit.** Here  $t_8 = t_9 = t_{10} = t_{11} = 0$  and hence all obstruction relations are satisfied. Therefore we find that for all  $t_1, t_2, t_3, t_4, t_5, t_6, t_7$ , the following deformation is integrable:

$$\begin{aligned} \varphi &= t_1(c_1 + c_7) + t_2(c_1 - c_7) + t_3c_3 + t_4c_5 + t_5(c_2 - c_8) + t_6c_4 + t_7c_6 \\ &= (t_1 + t_2 - it_5)c_1 + (t_3 - it_6)c_2 + (t_4 - it_7)c_3 + (t_1 - t_2 + it_5)c_4 + \text{c.c.} \end{aligned} \quad (41)$$

The corresponding brackets are

$$\begin{aligned} [R, B_a] &= \epsilon_{ab}B_b & [H, B_a] &= (t_1 + t_2)B_a + t_5\epsilon_{ab}B_b + t_3P_a + t_6\epsilon_{ab}P_b \\ [R, P_a] &= \epsilon_{ab}P_b & [H, P_a] &= t_4B_a + t_7\epsilon_{ab}B_b + (t_1 - t_2)P_a - t_5\epsilon_{ab}P_b, \end{aligned} \quad (42)$$

or in complex form

$$\begin{aligned} [R, \mathbf{B}] &= -i\mathbf{B} & [H, \mathbf{B}] &= (t_1 + t_2 - it_5)\mathbf{B} + (t_3 - it_6)\mathbf{P} \\ [R, \mathbf{P}] &= -i\mathbf{P} & [H, \mathbf{P}] &= (t_4 - it_7)\mathbf{B} + (t_1 - t_2 + it_5)\mathbf{P}. \end{aligned} \quad (43)$$



We may now use  $G$  (which we have not used yet, since the zero vector has all of  $G$  as stabiliser) to bring the bracket to a normal form. Recall that  $G$  acts as general linear transformations in  $\mathbf{B}$  and  $\mathbf{P}$  and by rescaling  $H$  by a nonzero real number. The adjoint action of  $H$  on  $\mathbf{B}$  and  $\mathbf{P}$  is defined by the matrix

$$M_H = \begin{pmatrix} t_1 + t_2 - it_5 & t_4 - it_7 \\ t_3 it_6 & t_1 - t_2 + it_5 \end{pmatrix}, \quad (44)$$

so that under  $G$

$$M_H \mapsto \lambda A^{-1} M_H A \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{C}^2) \quad (45)$$

and  $\lambda \in \mathbb{R}^\times$ . In other words, we can conjugate  $M_H$  and multiply it by a real scale.

Let us first focus on conjugation, which does not change the trace, which we see from the explicit form of  $M_H$  that it is real and equal to  $2t_1$ . A complex  $2 \times 2$  matrix is either diagonalisable or not. If diagonalisable, it may be conjugated to a diagonal matrix, which, if it has real trace, must take the form

$$\begin{pmatrix} \mu_1 + i\theta & 0 \\ 0 & \mu_2 - i\theta \end{pmatrix} \quad \text{for some } \mu_i, \theta \in \mathbb{R}. \quad (46)$$

Moreover, by relabelling  $\mathbf{B}$  and  $\mathbf{P}$ , if necessary, we can assume that  $\mu_1 \geq \mu_2$ . If  $M_H$  is not diagonalisable, then we can bring it to a Jordan form,

$$\begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix} \quad \text{for some } \nu \in \mathbb{R}, \text{ for the trace to be real.} \quad (47)$$

We distinguish several cases.

6.1.1.  $M_H$  *diagonalisable with*  $\mu_1 = \mu_2 = 0$ . To have a deformation at all, it must be that  $\theta \neq 0$ . In that case,  $\tilde{H} := \frac{1}{2\theta}(H - \theta R)$  obeys  $[\tilde{H}, \mathbf{B}] = i\mathbf{B}$  and  $[\tilde{H}, \mathbf{P}] = 0$ . In summary, the deformation can be brought to the complex form

$$[\mathbf{H}, \mathbf{B}] = i\mathbf{B}, \quad (48)$$

or to the real form

$$[\mathbf{H}, \mathbf{B}_a] = -\epsilon_{ab} \mathbf{B}_b. \quad (49)$$

Although it may not look it, this Lie algebra is isomorphic to the **euclidean Newton** algebra, whose real form is typically given by

$$[\mathbf{H}, \mathbf{B}_a] = \mathbf{P}_a \quad \text{and} \quad [\mathbf{H}, \mathbf{P}_a] = -\mathbf{B}_a. \quad (50)$$

Indeed, defining  $\mathbf{B}'_a = \mathbf{B}_a - \epsilon_{ab} \mathbf{P}_b$ ,  $\mathbf{P}'_a = \mathbf{B}_a + \epsilon_{ab} \mathbf{P}_b$  and  $\mathbf{H}' = -\frac{1}{2}(\mathbf{H} + \mathbf{R})$ , we see that the standard Newton algebra brackets imply that the primed generators obey the Lie brackets in equation (49).

6.1.2.  $M_H$  *diagonalisable with*  $0 \neq \mu_1 \geq \mu_2$ . In this case  $\tilde{H} = \frac{1}{\mu_1}(\mathbf{H} + \theta \mathbf{R})$  satisfies  $[\tilde{H}, \mathbf{B}] = \mathbf{B}$  and  $[\tilde{H}, \mathbf{P}] = (\lambda - 2i\theta')\mathbf{P}$ , where  $\lambda = \mu_2/\mu_1 \leq 1$ . If  $\lambda < -1$ , we can let  $\tilde{H} = \frac{1}{\mu_2}(\mathbf{H} + \theta \mathbf{R})$  instead and exchanging  $\mathbf{B}$  and  $\mathbf{P}$ , so that in any case we can bring the deformation to the complex form

$$[\mathbf{H}, \mathbf{B}] = \mathbf{B} \quad \text{and} \quad [\mathbf{H}, \mathbf{P}] = (\lambda + i\theta)\mathbf{P} \quad \text{where } \lambda \in [-1, 1] \text{ and } \theta \in \mathbb{R}. \quad (51)$$

Moreover we can always assume that  $\theta \geq 0$ , for if  $\theta < 0$ , then define  $\mathbf{H}' = \mathbf{H} + 2\theta \mathbf{R}$ ,  $\mathbf{B}' = \mathbf{P}$  and  $\mathbf{P}' = \mathbf{B}$  and we arrive at the the same algebra where  $\theta$  has become  $-\theta$ . In summary,

$$[\mathbf{H}, \mathbf{B}] = \mathbf{B} \quad \text{and} \quad [\mathbf{H}, \mathbf{P}] = (\lambda + i\theta)\mathbf{P} \quad \text{where } \lambda \in [-1, 1] \text{ and } \theta \geq 0, \quad (52)$$

or in real form

$$[\mathbf{H}, \mathbf{B}_a] = \mathbf{B}_a \quad \text{and} \quad [\mathbf{H}, \mathbf{P}_a] = \lambda \mathbf{P}_a - \epsilon_{ab} \theta \mathbf{P}_b. \quad (53)$$

The case  $\lambda = -1$  and  $\theta = 0$  is the **lorentzian Newton** algebra.

6.1.3.  $M_H$  *nondiagonalisable with nonzero trace*. Since  $M_H$  is not diagonalisable, its normal form is given by (47) with  $\nu \neq 0$  for nonzero trace. By rescaling we can bring the trace to any desired nonzero value, so we may as well take  $\nu = 1$  in (47) and arrive at the Lie brackets in complex form

$$[\mathbf{H}, \mathbf{B}] = \mathbf{B} \quad \text{and} \quad [\mathbf{H}, \mathbf{P}] = \mathbf{B} + \mathbf{P}, \quad (54)$$

and in real form

$$[\mathbf{H}, \mathbf{B}_a] = \mathbf{B}_a \quad \text{and} \quad [\mathbf{H}, \mathbf{P}_a] = \mathbf{B}_a + \mathbf{P}_a. \quad (55)$$

6.1.4.  $M_H$  *nondiagonalisable with zero trace*. In this case,  $M_H$  can be conjugated to (47) with  $v = 0$ , leading to the nonzero Lie brackets

$$[H, P] = B, \quad (56)$$

which is isomorphic to the **galilean algebra**. Usually one relabels  $B$  and  $P$  and writes the algebra as

$$[H, B_a] = P_a. \quad (57)$$

6.2. **Lightlike orbit**. Here  $t_8 = t_{10} = t_{11} = 0$  and  $t_9 = 1$ . The obstruction relations (40) are equivalent to  $t_1 = t_2 = t_4 = t_7 = 0$ . This leaves the following deformation

$$\begin{aligned} \varphi &= t_3 c_3 + t_5 (c_2 - c_8 + c_9) + t_6 c_4 + c_{10} \\ &= (t_3 - it_6) c_2 + t_5 (-ic_1 + ic_4 - c_5) - c_6 + c.c \end{aligned} \quad (58)$$

In order to bring this to a normal form, it is convenient to use the subgroup of  $G$  which stabilises the vector defining the lightlike orbit to bring parameters in another  $G$ -submodule of  $\mathcal{H}^2$  to a simpler form. In the basis  $(c_{10}, ic_8, -ic_8, c_6)$ , the lightlike vector labeling this orbit has components  $(0, 0, 0, -2)$ . From equation (22) we can easily determine that the subgroup of  $G$  which stabilises this vector is given by

$$G_{\text{lightlike}} = \left\{ \left( \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, |a|^2 \right) \in GL(\mathbb{C}^2) \times \mathbb{R}^\times \right\}. \quad (59)$$

From equation (21), we see that a typical element  $(A, \lambda) \in G$  acts on the complex vector subspace spanned by  $(c_1 - c_4, c_2, c_3)$  via

$$\frac{1}{\lambda \Delta} \begin{pmatrix} ad + bc & bd & -ac \\ 2cd & d^2 & -c^2 \\ -2ab & -b^2 & a^2 \end{pmatrix}, \quad (60)$$

so that a typical element of  $G_{\text{lightlike}}$  acts like

$$\frac{1}{|a|^2} \begin{pmatrix} 1 & 0 & -\frac{c}{d} \\ 2\frac{c}{a} & \frac{d}{a} & -\frac{c^2}{ad} \\ 0 & 0 & \frac{a}{d} \end{pmatrix}. \quad (61)$$

The component of the deformation  $\varphi$  in that three-dimensional subspace is parametrised by  $(-it_5, t_3 - it_6, 0)$ , which transforms under  $G_{\text{lightlike}}$  as

$$\begin{pmatrix} -it_5 \\ t_3 - it_6 \\ 0 \end{pmatrix} \mapsto \frac{1}{|a|^2} \begin{pmatrix} -it_5 \\ -2i\frac{c}{a}t_5 + \frac{d}{a}(t_3 - it_6) \\ 0 \end{pmatrix}. \quad (62)$$

We distinguish several branches.

6.2.1.  $t_5 = 0$  and  $t_3 - it_6 = 0$  *branch*. Here  $\varphi = -2c_6 = c_{10}$  and the deformation has additional nonzero Lie brackets  $[B_a, B_b] = \epsilon_{ab} H$ . Rescaling  $H$  and using the complex description, we may write this deformation as

$$[B, \bar{B}] = iH, \quad (63)$$

which is isomorphic (by rescaling  $H$  back) to the following real form

$$[B_a, B_b] = \epsilon_{ab} H. \quad (64)$$

6.2.2.  $t_5 = 0$  and  $t_3 - it_6 \neq 0$  *branch*. Here  $t_3 - it_6$  can be brought to 1, so that the deformation has additional nonzero Lie brackets

$$[H, B_a] = P_a \quad \text{and} \quad [B_a, B_b] = \epsilon_{ab} H. \quad (65)$$

After rescaling  $H$  and  $P$  and in a complex basis, we arrive at

$$[H, B] = P \quad \text{and} \quad [B, \bar{B}] = iH. \quad (66)$$

6.2.3.  $t_5 \neq 0$  branch. Here we can bring  $t_3 - it_6$  to zero and  $t_5$  to  $\pm 1$ , resulting in the Lie brackets

$$[H, B_a] = \pm \epsilon_{ab} B_b \quad [H, P_a] = \mp \epsilon_{ab} P_b \quad [B_a, B_b] = \epsilon_{ab} (H \pm R). \quad (67)$$

Finally, by redefining  $H \pm R$  to be the new  $H$  and after rescaling  $B$  and the new  $H$ , we may bring these brackets to the following complex form

$$[H, B] = \pm iB \quad \text{and} \quad [B, \bar{B}] = iH, \quad (68)$$

which is isomorphic to

$$[H, B_a] = \pm \epsilon_{ab} B_b \quad \text{and} \quad [B_a, B_b] = \epsilon_{ab} H. \quad (69)$$

6.3. **Timelike orbit.** Here  $t_8 = t_{11} = 0$ ,  $t_9 = 1$  and  $t_{10} = -1$ . The obstruction relations (40) are equivalent to  $t_1 = t_2 = 0$ ,  $t_4 = -t_3$  and  $t_7 = t_6$ . This leaves the following deformation

$$\begin{aligned} \varphi &= t_3(c_3 - c_5 + c_{11}) + t_5(c_2 - c_8 + c_9 - c_{15}) + t_6(c_4 + c_6 + c_{12}) + c_{10} + c_{16} \\ &= (t_3 - it_6)(c_2 + 2c_7) - (t_3 + it_6)c_3 + t_5(-ic_1 + ic_4 - c_5 + c_9) - c_6 - c_{10} + c.c \end{aligned} \quad (70)$$

In the basis  $(c_{10}, i\bar{c}_8, -ic_8, c_6)$ , the timelike vector labeling this orbit has components  $(-2, 0, 0, -2)$ . From equation (22) we can easily determine that the subgroup of  $G$  which stabilises this vector is given by

$$G_{\text{timelike}} = \left\{ \left( \begin{pmatrix} a & b \\ -\gamma\bar{b} & \gamma\bar{a} \end{pmatrix}, |a|^2 + |b|^2 \right) \in GL(\mathbb{C}^2) \times \mathbb{R}^\times \mid \gamma \in \mathbb{C} \quad |\gamma| = 1 \right\}. \quad (71)$$

From equation (21), we see that a typical element  $(A, \lambda) \in G$  acts on the complex vector subspace spanned by  $(c_1 - c_4, c_2, c_3)$  via

$$\frac{1}{\lambda\Delta} \begin{pmatrix} ad + bc & bd & -ac \\ 2cd & d^2 & -c^2 \\ -2ab & -b^2 & a^2 \end{pmatrix}, \quad (72)$$

so that a typical element of  $G_{\text{timelike}}$  acts like

$$\frac{1}{(|a|^2 + |b|^2)^2} \begin{pmatrix} |a|^2 - |b|^2 & \bar{a}b & a\bar{b} \\ -2\gamma\bar{a}\bar{b} & \gamma\bar{a}^2 & -\gamma\bar{b}^2 \\ -2\gamma ab & -\bar{\gamma}b^2 & \bar{\gamma}a^2 \end{pmatrix}. \quad (73)$$

The component of the deformation  $\varphi$  in that three-dimensional subspace is parametrised by  $(-it_5, t_3 - it_6, -t_3 - it_6)$ , which transforms under  $G_{\text{timelike}}$  as

$$\begin{pmatrix} -it_5 \\ t_3 - it_6 \\ -t_3 - it_6 \end{pmatrix} \mapsto \frac{1}{(|a|^2 + |b|^2)^2} \begin{pmatrix} -i(|a|^2 - |b|^2)t_5 + \bar{a}b(t_3 - it_6) - a\bar{b}(t_3 + it_6) \\ 2i\gamma\bar{a}\bar{b}t_5 + \gamma\bar{a}^2(t_3 - it_6) + \gamma\bar{b}^2(t_3 + it_6) \\ 2i\bar{\gamma}abt_5 - \bar{\gamma}b^2(t_3 - it_6) - \bar{\gamma}a^2(t_3 + it_6) \end{pmatrix}. \quad (74)$$

Acting on  $(t_5, t_3, t_6)$  we have

$$\begin{pmatrix} t_5 \\ t_3 \\ t_6 \end{pmatrix} \mapsto \frac{1}{(|a|^2 + |b|^2)^2} \begin{pmatrix} |a|^2 - |b|^2 & -2\text{Im}(\bar{a}b) & 2\text{Re}(\bar{a}b) \\ -2\text{Im}(\gamma\bar{a}\bar{b}) & \text{Re}(\gamma(\bar{a}^2 + \bar{b}^2)) & \text{Im}(\gamma(\bar{a}^2 - \bar{b}^2)) \\ -2\text{Re}(\gamma\bar{a}\bar{b}) & -\text{Im}(\gamma(\bar{a}^2 + \bar{b}^2)) & \text{Re}(\gamma(\bar{a}^2 - \bar{b}^2)) \end{pmatrix} \begin{pmatrix} t_5 \\ t_3 \\ t_6 \end{pmatrix}. \quad (75)$$

The kernel of this representation consists of those matrices with  $|a| = 1$ ,  $\gamma = a^2$  and  $b = 0$ , which is a circle subgroup. Therefore the action is not faithful and only a 4-dimensional subgroup of  $G_{\text{timelike}}$  acts effectively on  $(t_5, t_3, t_6)$ . We observe that the matrix in equation (75) is conformally orthogonal; that is, if we let

$$M := \frac{1}{(|a|^2 + |b|^2)^2} \begin{pmatrix} |a|^2 - |b|^2 & -2\text{Im}(\bar{a}b) & 2\text{Re}(\bar{a}b) \\ -2\text{Im}(\gamma\bar{a}\bar{b}) & \text{Re}(\gamma(\bar{a}^2 + \bar{b}^2)) & \text{Im}(\gamma(\bar{a}^2 - \bar{b}^2)) \\ -2\text{Re}(\gamma\bar{a}\bar{b}) & -\text{Im}(\gamma(\bar{a}^2 + \bar{b}^2)) & \text{Re}(\gamma(\bar{a}^2 - \bar{b}^2)) \end{pmatrix}, \quad (76)$$

then  $M^T M = (|a|^2 + |b|^2)^{-2} \mathbb{1}$ . We wish to conclude that  $G_{\text{timelike}}$  acts on the three-dimensional space with coordinates  $(t_5, t_3, t_6)$  in such a way that there are two orbits: the zero vector and all the nonzero vectors. Since the action is linear, it is clear that the zero vector is its own orbit, so what we need to show is that all nonzero vectors lie on the same orbit. It is enough to show that the orbit of, say, the vector  $(1, 0, 0)$  under the orthogonal matrices  $(|a|^2 + |b|^2)M$ , as  $a, b, \gamma$  vary, is all of the unit sphere. If we write  $a = ue^{i\theta}$ ,  $b = ve^{i\psi}$  and  $\gamma = e^{i\phi}$ , then the image of  $(1, 0, 0)$  under the matrix  $(u^2 + v^2)M$  is given by

$$\left( \frac{u^2 - v^2}{u^2 + v^2}, \frac{2uv}{u^2 + v^2} \cos(\phi - \theta - \psi), \frac{-2uv}{u^2 + v^2} \sin(\phi - \theta - \psi) \right), \quad (77)$$

and if we let  $\phi - \theta - \psi = -\vartheta$  and introduce  $\rho = -v/u$  (assuming  $u \neq 0$ , which is the pole  $(-1, 0, 0)$  corresponding to the point at infinity), then the above vector becomes

$$\left( \frac{1 - \rho^2}{1 + \rho^2}, \frac{2\rho}{1 + \rho^2} \cos \vartheta, \frac{2\rho}{1 + \rho^2} \sin \vartheta \right), \quad (78)$$

which we recognise as the stereographic projection which parametrises the unit sphere (minus a pole) in terms of the complex numbers  $\rho e^{i\vartheta}$ , up to a relabelling of the coordinates. Therefore the action of  $G_{\text{timelike}}$  is as claimed and hence acting with  $G_{\text{timelike}}$  we can bring  $(t_5, t_3, t_6)$  to one of two canonical forms:  $(0, 0, 0)$  or  $(1, 0, 0)$ , which leads to two different deformations.

6.3.1. *(0, 0, 0) normal form.* If  $t_3 = t_5 = t_6 = 0$ , then the deformation is simply  $\varphi = -2c_6 - 2c_{10}$ , so that rescaling  $H$  we can bring the Lie brackets to

$$[\mathbf{B}, \bar{\mathbf{B}}] = iH \quad \text{and} \quad [\mathbf{P}, \bar{\mathbf{P}}] = iH, \quad (79)$$

which is isomorphic to the following

$$[B_a, B_b] = \epsilon_{ab} H \quad \text{and} \quad [P_a, P_b] = \epsilon_{ab} H. \quad (80)$$

6.3.2. *(1, 0, 0) normal form.* On the other hand, if  $t_3 = t_6 = 0$  and  $t_5 = 1$ , the deformation becomes

$$\begin{aligned} \varphi &= c_2 - c_8 + c_9 - c_{15} + c_{10} + c_{16} \\ &= -i(c_1 - \bar{c}_1) + i(c_4 - \bar{c}_4) - 2c_5 + 2c_9 - 2c_6 - 2c_{10}, \end{aligned} \quad (81)$$

which leads to the Lie brackets

$$[H, \mathbf{B}] = -i\mathbf{B}, \quad [H, \mathbf{P}] = i\mathbf{P}, \quad [\mathbf{B}, \bar{\mathbf{B}}] = -2i(H + R) \quad \text{and} \quad [\mathbf{P}, \bar{\mathbf{P}}] = -2i(H - R). \quad (82)$$

If we let  $H \mapsto -\frac{1}{2}(R + H)$  and rescale both  $\mathbf{B}$  and  $\mathbf{P}$  by a factor of  $\frac{1}{2}$ , we arrive at the following Lie brackets

$$[H, \mathbf{B}] = i\mathbf{B}, \quad [\mathbf{B}, \bar{\mathbf{B}}] = iH \quad \text{and} \quad [\mathbf{P}, \bar{\mathbf{P}}] = i(H + R), \quad (83)$$

which is isomorphic to

$$[H, B_a] = \epsilon_{ab} B_b, \quad [B_a, B_b] = \epsilon_{ab} H \quad \text{and} \quad [P_a, P_b] = \epsilon_{ab}(H - R). \quad (84)$$

6.4. **Spacelike orbit.** In this case  $t_8 = 1$ , but  $t_9 = t_{10} = t_{11} = 0$ . The obstruction relations (40) are equivalent to  $t_1 = t_3 = t_4 = t_5 = 0$ . This leaves the following deformation

$$\begin{aligned} \varphi &= t_2(c_1 - c_7 + c_{11}) + t_6(c_4 + c_9) + t_7(c_6 + c_{15}) + c_{14} \\ &= t_2(c_1 - c_4 + 2c_7) - t_6(c_5 + ic_2) - t_7(c_9 + ic_3) - 2ic_8 + \text{c.c.} \end{aligned} \quad (85)$$

Relative to the ordered basis  $(c_{10}, i\bar{c}_8, -ic_8, c_6)$  the vector labelling this orbit has components  $(0, 2, 2, 0)$ . Using (22) we can determine the stabiliser  $G_{\text{spacelike}}$  of this vector and we find that it consists of the union (not disjoint)

$$G_{\text{spacelike}} = G' \cup G'' \quad (86)$$

of the two subsets of  $G$  defined by

$$G' = \left\{ \left( z \begin{pmatrix} 1 & is \\ itu & u \end{pmatrix}, |z|^2 u(1 + st) \right) \in GL(\mathbb{C}^2) \times \mathbb{R}^\times \mid z \in \mathbb{C}^\times, s, t \in \mathbb{R}, st \neq -1, u \in \mathbb{R}^\times \right\} \quad (87)$$

and

$$G'' = \left\{ \left( z \begin{pmatrix} is & 1 \\ u & itu \end{pmatrix}, |z|^2 u(1 + st) \right) \in GL(\mathbb{C}^2) \times \mathbb{R}^\times \mid z \in \mathbb{C}^\times, s, t \in \mathbb{R}, st \neq -1, u \in \mathbb{R}^\times \right\}. \quad (88)$$

Using (21) we can determine how  $G_{\text{spacelike}}$  acts on the three-dimensional real subspace with ordered basis  $(c_1 - c_4, -ic_2, -ic_3)$ . The component of  $\varphi$  in this subspace has coordinates  $(t_2, t_6, t_7)$  and we find that under a typical element of  $G' \subset G_{\text{spacelike}}$

$$\begin{pmatrix} t_2 \\ t_6 \\ t_7 \end{pmatrix} \mapsto \frac{1}{u^2(1 + st)^2|z|^2} \begin{pmatrix} u(1 - st) & us & -ut \\ -2tu^2 & u^2 & u^2 t^2 \\ 2s & s^2 & 1 \end{pmatrix} \begin{pmatrix} t_2 \\ t_6 \\ t_7 \end{pmatrix}; \quad (89)$$

whereas under a typical element of  $G'' \subset G_{\text{spacelike}}$

$$\begin{pmatrix} t_2 \\ t_6 \\ t_7 \end{pmatrix} \mapsto \frac{1}{u^2(1 + st)^2|z|^2} \begin{pmatrix} -u(1 - st) & -ut & us \\ 2tu^2 & u^2 t^2 & u^2 \\ -2s & 1 & s^2 \end{pmatrix} \begin{pmatrix} t_2 \\ t_6 \\ t_7 \end{pmatrix}. \quad (90)$$

This action is conformally orthogonal relative to a lorentzian inner product on this three-dimensional space. Indeed, if we transform  $(t_2, t_6, t_7)^T$  by either of the two matrices below (rescaled versions of the matrices in  $G'$  and  $G''$ , respectively),

$$\frac{1}{u(1+st)} \begin{pmatrix} u(1-st) & us & -ut \\ -2tu^2 & u^2 & u^2t^2 \\ 2s & s^2 & 1 \end{pmatrix} \quad \frac{1}{u(1+st)} \begin{pmatrix} -u(1-st) & -ut & us \\ 2tu^2 & u^2t^2 & u^2 \\ -2s & 1 & s^2 \end{pmatrix} \quad (91)$$

we find that the indefinite quadratic form  $t_2^2 - t_6t_7$  is invariant. Therefore, acting with the matrices in either  $G'$  or  $G''$  above, the quadratic form is rescaled by a positive factor  $u^{-2}(1+st)^{-2}|z|^{-4}$ . The determinant of the matrices in either  $G'$  or  $G''$  above is given by  $u^{-3}(1+st)^{-3}|z|^{-6}$ , which can be either positive or negative. Consider acting on a vector with coordinates  $(0, t_6, t_7)$ . Under  $G'$  or  $G''$ , respectively, this vector is sent to

$$\begin{pmatrix} t_2' \\ t_6' \\ t_7' \end{pmatrix} := \frac{1}{u^2(1+st)^2|z|^2} \begin{pmatrix} u(st_6 - tt_7) \\ u^2t_6 + u^2t^2t_7 \\ s^2t_6 + t_7 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} t_2'' \\ t_6'' \\ t_7'' \end{pmatrix} := \frac{1}{u^2(1+st)^2|z|^2} \begin{pmatrix} u(tt_6 - st_7) \\ u^2t^2t_6 + u^2t_7 \\ t_6 + s^2t_7 \end{pmatrix} \quad (92)$$

and therefore if  $t_6$  and  $t_7$  are positive (resp. negative) so will be  $t_6'$ ,  $t_7'$ ,  $t_6''$  and  $t_7''$ . In other words  $G_{\text{spacelike}}$  preserves the time orientation. This means that the action of  $G_{\text{spacelike}}$  on the three-dimensional space spanned by  $(t_2, t_6, t_7)$  defines a homomorphism  $G_{\text{spacelike}} \rightarrow \text{CO}(2, 1)_+$  whose kernel consists of elements of the form

$$\left( \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, 1 \right) \quad \text{with } |z| = 1. \quad (93)$$

By dimension count, the Lie algebra homomorphism  $\mathfrak{g}_{\text{spacelike}} \rightarrow \mathfrak{co}(2, 1)$  is surjective and therefore, by the Lie correspondence for linear groups (which these clearly are), it induces a surjective group homomorphism from the identity component of  $G_{\text{spacelike}}$  to that of  $\text{CO}(2, 1)_+$ , which is the proper orthochronous conformal Lorentz group. That group, and hence also  $G_{\text{spacelike}}$ , acts on the three-dimensional space of vectors  $\mathbf{t} = (t_2, t_6, t_7)$  with the following six orbits, labelled by the given vector  $\mathbf{t}$ :

- (1) **zero orbit**, with  $\mathbf{t} = (0, 0, 0)$ ;
- (2) **spacelike orbit**, with  $\mathbf{t} = (1, 0, 0)$ ;
- (3) **past and future lightlike orbits**, with  $\mathbf{t} = \pm(0, 0, 1)$ ; and
- (4) **past and future timelike orbits**, with  $\mathbf{t} = \pm(0, 1, 1)$ .

We shall now consider the deformations corresponding to these six orbits.

**6.4.1. Zero orbit.** In this case  $t_2 = t_6 = t_7 = 0$  and hence  $\varphi = -2i\mathbf{c}_8 + \text{c.c.}$ , which leads (after rescaling  $H$ ) to the Lie brackets  $[\mathbf{B}, \bar{\mathbf{P}}] = iH$ . However we may simply rotate  $\mathbf{P}$  and reabsorb the  $i$  and write this Lie algebra as

$$\boxed{[\mathbf{B}, \bar{\mathbf{P}}] = H}, \quad (94)$$

which is isomorphic to

$$[\mathbf{B}_a, \mathbf{P}_b] = \delta_{ab}H. \quad (95)$$

This is the **Carroll** algebra.

**6.4.2. Spacelike orbit.** In this case  $t_2 = 1$  and  $t_6 = t_7 = 0$ , so that

$$\varphi = \mathbf{c}_1 - \mathbf{c}_4 + 2\mathbf{c}_7 - 2i\mathbf{c}_8 + \text{c.c.}, \quad (96)$$

which leads to the following Lie brackets

$$\boxed{[\mathbf{H}, \mathbf{B}] = \mathbf{B}, \quad [\mathbf{H}, \mathbf{P}] = -\mathbf{P} \quad \text{and} \quad [\mathbf{B}, \bar{\mathbf{P}}] = 2(\mathbf{R} - i\mathbf{H})}, \quad (97)$$

which is isomorphic to

$$[\mathbf{H}, \mathbf{B}_a] = \mathbf{B}_a, \quad [\mathbf{H}, \mathbf{P}_a] = -\mathbf{P}_a \quad \text{and} \quad [\mathbf{B}_a, \mathbf{P}_b] = \delta_{ab}\mathbf{R} + \epsilon_{ab}\mathbf{H}. \quad (98)$$

This is isomorphic to  $\mathfrak{so}(3, 1)$ , which we think of as the de Sitter (or hyperbolic) algebra in 2+1 dimensions.

6.4.3. *Lightlike orbits.* In this case  $t_2 = t_6 = 0$  and  $t_7 = \tau$ , where  $\tau = \pm 1$ . The deformation is

$$\varphi = -\tau(\mathbf{c}_9 + i\mathbf{c}_3) - 2i\mathbf{c}_8 + \text{c.c.}, \quad (99)$$

which leads to the following Lie brackets (after multiplying  $\mathbf{P}$  by  $-i$ )

$$\boxed{[\mathbf{H}, \mathbf{P}] = \tau\mathbf{B}, \quad [\mathbf{B}, \bar{\mathbf{P}}] = -2\mathbf{H}, \quad \text{and} \quad [\mathbf{P}, \bar{\mathbf{P}}] = -2\tau i\mathbf{R}}, \quad (100)$$

which is isomorphic to

$$[\mathbf{H}, \mathbf{P}_a] = \tau\mathbf{B}_a, \quad [\mathbf{B}_a, \mathbf{P}_b] = -\delta_{ab}\mathbf{H} \quad \text{and} \quad [\mathbf{P}_a, \mathbf{P}_b] = \tau\epsilon_{ab}\mathbf{R}. \quad (101)$$

These are isomorphic to the euclidean algebra  $\mathfrak{e}$  for  $\tau = 1$  and the Poincaré algebra  $\mathfrak{p}$  for  $\tau = -1$ .

6.4.4. *Timelike orbits.* In this case  $t_2 = 0$  and  $t_6 = t_7 = \tau$ , where  $\tau = \pm 1$ . The deformation in this case is

$$\varphi = -\tau(i\mathbf{c}_2 + i\mathbf{c}_3 + \mathbf{c}_5 + \mathbf{c}_9) - 2i\mathbf{c}_8 + \text{c.c.}, \quad (102)$$

with corresponding Lie brackets given by (after multiplying  $\mathbf{P}$  by  $-i$  and  $\mathbf{H}$  by  $\tau$ ),

$$\boxed{[\mathbf{H}, \mathbf{B}] = -\mathbf{P}, \quad [\mathbf{H}, \mathbf{P}] = \mathbf{B}, \quad [\mathbf{B}, \bar{\mathbf{B}}] = -2\tau i\mathbf{R}, \quad [\mathbf{B}, \bar{\mathbf{P}}] = -2\tau\mathbf{H} \quad \text{and} \quad [\mathbf{P}, \bar{\mathbf{P}}] = -2\tau i\mathbf{R}}, \quad (103)$$

which is isomorphic to

$$\begin{aligned} [\mathbf{H}, \mathbf{B}_a] &= -\mathbf{P}_a & [\mathbf{B}_a, \mathbf{B}_b] &= \tau\epsilon_{ab}\mathbf{R} & [\mathbf{B}_a, \mathbf{P}_b] &= -\tau\delta_{ab}\mathbf{H}. \\ [\mathbf{H}, \mathbf{P}_a] &= \mathbf{B}_a & [\mathbf{P}_a, \mathbf{P}_b] &= \tau\epsilon_{ab}\mathbf{R} \end{aligned} \quad (104)$$

This is isomorphic to  $\mathfrak{so}(4)$  for  $\tau = 1$  or  $\mathfrak{so}(2, 2)$  for  $\tau = -1$ . We can rescale  $\mathbf{B}$  and  $\mathbf{P}$  in equation (103) in order to eliminate the factors of 2 from the last three brackets in the complex form of the algebra, but this reintroduces some factors of 2 in the real form of the algebra.

## 7. INVARIANT INNER PRODUCTS

Recall that a Lie algebra  $\mathfrak{k}$  is said to be **metric**, if  $\mathfrak{k}$  admits a nondegenerate symmetric bilinear form  $(-, -) : \mathfrak{k} \times \mathfrak{k} \rightarrow \mathbb{R}$  satisfying the “associativity” condition:

$$([x, y], z) = (x, [y, z]) \quad \forall x, y, z \in \mathfrak{k}. \quad (105)$$

The Killing form  $\kappa(x, y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y)$  is always associative, but Cartan’s semisimplicity criterion says that it is only nondegenerate for semisimple Lie algebras. Among the kinematical Lie algebras found above (and summarised in Table 1), only  $\mathfrak{so}(3, 1)$ ,  $\mathfrak{so}(4)$  and  $\mathfrak{so}(2, 2)$  are semisimple and therefore metric. However there are also non-simple kinematical Lie algebras in the table which are metric. In this section we investigate the metricity of the non-simple kinematical Lie algebras in Table 1. The results are summarised in the right-most column of that table.

In determining whether or not a kinematical Lie algebra is metric, it is more convenient to work with the real form of the Lie algebra. The strategy in many cases is simply to exploit the associativity condition (105) to conclude that no invariant inner product exists.

To show that the static kinematical Lie algebra (5) does not admit an invariant inner product, let  $X$  be any of  $\mathbf{B}$  or  $\mathbf{P}$ . Then, for any associative symmetric bilinear form,

$$\epsilon_{ab}(\mathbf{B}_b, \mathbf{X}_c) = ([\mathbf{R}, \mathbf{B}_a], \mathbf{X}_c) = (\mathbf{R}, [\mathbf{B}_a, \mathbf{X}_c]) = 0. \quad (106)$$

Since  $\mathbf{B}$  can only have nonzero inner product with  $\mathbf{B}$  or  $\mathbf{P}$ , we find that  $(\mathbf{B}_a, -) = 0$  and hence  $(-, -)$  is degenerate. That takes care of the first row in Table 1. The next five rows in Table 1 describe Lie algebras where the ideal spanned by  $\mathbf{B}$  and  $\mathbf{P}$  is abelian. The exact same argument as for the static kinematical Lie algebra shows that any associative symmetric bilinear form is degenerate. Finally, a similar argument shows that neither do the Lie algebras (63), (66) and (68), corresponding to the last three rows in Table 1, admit invariant inner products. Indeed, if  $(-, -)$  is an associative symmetric bilinear form, then if  $X$  stands for either  $\mathbf{B}$  or  $\mathbf{P}$ , we have

$$\epsilon_{ab}(\mathbf{P}_b, \mathbf{X}_c) = ([\mathbf{R}, \mathbf{P}_a], \mathbf{X}_c) = (\mathbf{R}, [\mathbf{P}_a, \mathbf{X}_c]) = 0, \quad (107)$$

so that  $(\mathbf{P}_a, -) = 0$ .

In this dimension, the Carroll, Poincaré and euclidean algebras are metric. The Carroll algebra (94) admits a two-parameter family of invariant inner products:

$$(\mathbf{B}_a, \mathbf{P}_b) = \epsilon_{ab}\lambda \quad (\mathbf{H}, \mathbf{R}) = \lambda \quad (\mathbf{R}, \mathbf{R}) = \mu \quad \forall \lambda, \mu \in \mathbb{R}, \quad \lambda \neq 0. \quad (108)$$

The euclidean algebra (100) ( $\tau = 1$ ) also admits a two-parameter family of invariant inner products:

$$(\mathbf{B}_a, \mathbf{P}_b) = \epsilon_{ab}\lambda \quad (\mathbf{P}_a, \mathbf{P}_b) = \delta_{ab}\mu \quad (\mathbf{H}, \mathbf{R}) = \lambda \quad (\mathbf{R}, \mathbf{R}) = \mu \quad \forall \lambda, \mu \in \mathbb{R}, \quad \lambda \neq 0, \quad (109)$$

and so does the Poincaré algebra (100) ( $\tau = -1$ ):

$$(B_a, P_b) = \epsilon_{ab}\lambda \quad (P_a, P_b) = -\delta_{ab}\mu \quad (H, R) = \lambda \quad (R, R) = \mu \quad \forall \lambda, \mu \in \mathbb{R}, \quad \lambda \neq 0. \quad (110)$$

Finally, the kinematical Lie algebras (79) and (83), which are unique to this dimension, are also metric. Indeed, the former algebra has a two-parameter family of invariant inner products given by

$$(B_a, B_b) = \delta_{ab}\lambda \quad (P_a, P_b) = \delta_{ab}\lambda \quad (H, R) = \lambda \quad (R, R) = \mu \quad \forall \lambda, \mu \in \mathbb{R}, \quad \lambda \neq 0, \quad (111)$$

and does the latter algebra, whose inner product is given by

$$(B_a, B_b) = \delta_{ab}\lambda \quad (P_a, P_b) = \delta_{ab}(\lambda - \mu) \quad (H, R) = \lambda \quad (H, H) = \lambda \quad (R, R) = \mu, \quad (112)$$

for all  $\lambda, \mu \in \mathbb{R}$  with  $\lambda \neq 0$  and  $\lambda \neq \mu$ .

## 8. SUMMARY

We have classified all kinematical real Lie algebras in dimension  $2+1$  (up to Lie algebra isomorphism) by classifying the deformations of the static kinematical Lie algebra, using the approach advocated in [4] and used in [6] to classify all kinematical Lie algebras in dimension  $D+1$  for  $D \geq 4$ . Since for  $D < 2$  the kinematical condition on a Lie algebra is vacuous, except for specifying the dimension, the results of this paper complete the classification of kinematical Lie algebras in any dimension. It should perhaps be remarked that in physical/geometrical applications, it is desirable to refine this classification and distinguish kinematical Lie algebras which, although isomorphic as Lie algebras, act differently on the  $(2+1)$ -dimensional spacetime. This finer classification is the subject of a forthcoming paper containing the classification of spacetimes for kinematical Lie algebras in all dimensions.

Table 1 displays the classification for  $D = 2$ . All Lie brackets are written in the complex form and share the brackets in equation (6), which are not written explicitly. We also have the Lie brackets obtained from the ones shown via complex conjugation, but we do not write them explicitly either. Thus the table contains the minimal data necessary to reconstruct the Lie algebras. In some cases, we have relabelled  $\mathbf{B}$  and  $\mathbf{P}$  in order to make the description more uniform. The Lie algebras below the line are unique to  $D = 2$ , whereas those above the line are  $D = 2$  versions of kinematical Lie algebras which occur also for any  $D > 2$ . In  $D = 3$  there are also some kinematical Lie algebras which have no analogue in any other dimension: there, due to the existence of the rotationally invariant vector product in  $\mathbb{R}^3$ , whereas the kinematical Lie algebras unique to  $D = 2$  owe their existence to the rotationally invariant symplectic structure on  $\mathbb{R}^2$ .

TABLE 1. Kinematical Lie algebras in  $2+1$  dimensions (complex form)

Eq.	Nonzero Lie brackets				Comments	Metric?
6					static	
56	$[H, B] = P$				galilean	
54	$[H, B] = B$	$[H, P] = B + P$				
52	$[H, B] = B$	$[H, P] = P$				
52	$[H, B] = B$	$[H, P] = (1 + i\theta)P$			$\theta > 0$	
52	$[H, B] = B$	$[H, P] = \lambda P$			$\lambda \in (-1, 1)$	
52	$[H, B] = B$	$[H, P] = -P$			lorentzian Newton	
48	$[H, B] = iB$			euclidean Newton		
94			$[B, \bar{P}] = 2H$		Carroll	✓(108)
100	$[H, P] = -B$		$[B, \bar{P}] = 2H$	$[P, \bar{P}] = -2iR$	$\epsilon$	✓(109)
100	$[H, P] = B$		$[B, \bar{P}] = 2H$	$[P, \bar{P}] = 2iR$	$p$	✓(110)
97	$[H, B] = B$	$[H, P] = -P$	$[B, \bar{P}] = 2(H + iR)$		$so(3, 1)$	✓
103	$[H, B] = P$	$[H, P] = -B$	$[B, \bar{B}] = -2iR$	$[B, \bar{P}] = 2H$	$so(4)$	✓
103	$[H, B] = -P$	$[H, P] = B$	$[B, \bar{B}] = 2iR$	$[B, \bar{P}] = 2H$	$so(2, 2)$	✓
52	$[H, B] = B$	$[H, P] = (\lambda + i\theta)P$			$\lambda \in (-1, 1)$ and $\theta > 0$	
79			$[B, \bar{B}] = iH$	$[P, \bar{P}] = iH$		✓(111)
83	$[H, B] = iB$	$[B, \bar{B}] = iH$		$[P, \bar{P}] = i(H + R)$		✓(112)
63			$[B, \bar{B}] = iH$			
66	$[H, B] = P$	$[B, \bar{B}] = iH$				
68	$[H, B] = \pm iB$	$[B, \bar{B}] = iH$				

The first six lines consist of Lie algebras which are the semidirect product of the abelian subalgebra generated by  $H$  and  $R$  and a four-dimensional real representation (real and imaginary parts of a two-dimensional complex representation spanned by  $B$  and  $P$ ), where representation where  $R$  acts as multiplication by  $-i$  and  $H$ , which commutes with  $R$  therefore acts complex linearly. This means that the action of  $H$  (relative to the basis  $B$  and  $P$ ) is characterised by a  $2 \times 2$  complex matrix. However not every such matrix gives rise to different (i.e., non-isomorphic) semidirect products. We can change basis  $(B, P) \mapsto (B', P')$ , which is the same as conjugating the matrix of  $H$  in  $GL(2, \mathbb{C})$ , but we can also modify



H itself by affine transformations of the form  $H \mapsto \lambda H + \mu R$ , where  $\lambda, \mu \in \mathbb{R}$  and  $\lambda \neq 0$ . The first six lines in the table correspond precisely to the isomorphism classes of such semidirect products.

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#### APPENDIX A. ENUMERATIONS OF THE DEFORMATION COMPLEX

In this appendix we enumerate the first few graded subspaces of the deformation complexes  $C^\bullet$  and  $C_\mathbb{C}^\bullet$ , which we will refer to informally as the real and complex deformations complexes.

**A.1. Enumeration of the real deformation complex.** We shall now enumerate bases for  $C^p$ ,  $p = 0, 1, 2, 3$ , and the dimension count in Section 2 will ensure that we have not left out any basis elements.  $C^0$  is spanned by  $R$  and  $H$ . Bases for  $C^1$ ,  $C^2$  and  $C^3$  are tabulated below in abbreviated form, where we distinguish between  $\beta\pi = \beta_a\pi_a$  and  $\epsilon\beta\pi = \epsilon_{ab}\beta_a\pi_b$ , et cetera. In particular, we can now have  $\epsilon\beta\beta = \epsilon_{ab}\beta_a\beta_b$ . Similarly, we must distinguish between  $\beta B = \beta_a B_a$  and  $\epsilon\beta B = \epsilon_{ab}\beta_a B_b$ . Notice however that for any  $X \in \mathfrak{g}$ ,  $\epsilon\beta\pi\beta X$  and  $\epsilon\beta\beta\pi X$  are collinear, et cetera. Similarly, any terms with two  $\epsilon$  can be rewritten with no  $\epsilon$ 's using the identity  $\epsilon_{ab}\epsilon_{cd} = \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad}$ .

TABLE 2. Basis for  $C^1(\mathfrak{h}; \mathfrak{g})^\tau$

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$
$\eta R$	$\eta H$	$\beta B$	$\epsilon\beta B$	$\beta P$	$\epsilon\beta P$	$\pi B$	$\epsilon\pi B$	$\pi P$	$\epsilon\pi P$

TABLE 3. Basis for  $C^2(\mathfrak{h}; \mathfrak{g})^\tau$

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$
$\eta\beta B$	$\eta\epsilon\beta B$	$\eta\beta P$	$\eta\epsilon\beta P$	$\eta\pi B$	$\eta\epsilon\pi B$	$\eta\pi P$	$\eta\epsilon\pi P$
$c_9$	$c_{10}$	$c_{11}$	$c_{12}$	$c_{13}$	$c_{14}$	$c_{15}$	$c_{16}$
$\frac{1}{2}\epsilon\beta\beta R$	$\frac{1}{2}\epsilon\beta\beta H$	$\beta\pi R$	$\epsilon\beta\pi R$	$\beta\pi H$	$\epsilon\beta\pi H$	$\frac{1}{2}\epsilon\pi\pi R$	$\frac{1}{2}\epsilon\pi\pi H$

TABLE 4. Basis for  $C^3(\mathfrak{h}; \mathfrak{g})^\tau$

$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$
$\eta\epsilon\beta\beta R$	$\eta\epsilon\beta\beta H$	$\eta\epsilon\pi\pi R$	$\eta\epsilon\pi\pi H$	$\eta\beta\pi R$	$\eta\epsilon\beta\pi R$	$\eta\beta\pi H$	$\eta\epsilon\beta\pi H$
$b_9$	$b_{10}$	$b_{11}$	$b_{12}$	$b_{13}$	$b_{14}$	$b_{15}$	$b_{16}$
$\epsilon\beta\beta\pi B$	$\epsilon\beta\beta\pi P$	$\epsilon\pi\pi\beta B$	$\epsilon\pi\pi\beta P$	$\beta\pi\beta B$	$\beta\pi\beta P$	$\beta\pi\pi B$	$\beta\pi\pi P$

Finally we work out the Nijenhuis–Richardson bracket  $C^2 \times C^2 \rightarrow C^3$ . Table 5 displays the multiplication table for  $\bullet : C^2 \times C^2 \rightarrow C^3$  from where we obtain the Nijenhuis–Richardson bracket by symmetrisation:

$\llbracket c_1, c_9 \rrbracket = b_1$	$\llbracket c_2, c_{16} \rrbracket = b_{15}$	$\llbracket c_5, c_9 \rrbracket = b_6$	$\llbracket c_7, c_{11} \rrbracket = b_5$
$\llbracket c_1, c_{10} \rrbracket = b_2$	$\llbracket c_3, c_{12} \rrbracket = b_1$	$\llbracket c_5, c_{10} \rrbracket = b_8 + \frac{1}{2}b_9$	$\llbracket c_7, c_{12} \rrbracket = b_6$
$\llbracket c_1, c_{11} \rrbracket = b_5$	$\llbracket c_3, c_{13} \rrbracket = b_{14}$	$\llbracket c_5, c_{12} \rrbracket = b_3$	$\llbracket c_7, c_{13} \rrbracket = b_7 + b_{16}$
$\llbracket c_1, c_{12} \rrbracket = b_6$	$\llbracket c_3, c_{14} \rrbracket = b_2 - \frac{1}{2}b_{10}$	$\llbracket c_5, c_{13} \rrbracket = b_{15}$	$\llbracket c_7, c_{14} \rrbracket = b_8 - \frac{1}{2}b_{12}$
$\llbracket c_1, c_{13} \rrbracket = b_7 + b_{13}$	$\llbracket c_3, c_{15} \rrbracket = b_6$	$\llbracket c_5, c_{14} \rrbracket = b_4 - \frac{1}{2}b_{11}$	$\llbracket c_7, c_{15} \rrbracket = b_3$
$\llbracket c_1, c_{14} \rrbracket = b_8 - \frac{1}{2}b_9$	$\llbracket c_3, c_{16} \rrbracket = b_8 + \frac{1}{2}b_{12}$	$\llbracket c_6, c_9 \rrbracket = b_5$	$\llbracket c_7, c_{16} \rrbracket = b_4$
$\llbracket c_1, c_{16} \rrbracket = \frac{1}{2}b_{11}$	$\llbracket c_4, c_{11} \rrbracket = -b_1$	$\llbracket c_6, c_{10} \rrbracket = b_7 - b_{13}$	$\llbracket c_8, c_{10} \rrbracket = -b_{14}$
$\llbracket c_2, c_{11} \rrbracket = b_6$	$\llbracket c_4, c_{13} \rrbracket = -b_2 + \frac{1}{2}b_{10}$	$\llbracket c_6, c_{11} \rrbracket = b_3$	$\llbracket c_8, c_{11} \rrbracket = -b_6$
$\llbracket c_2, c_{12} \rrbracket = -b_5$	$\llbracket c_4, c_{14} \rrbracket = b_{14}$	$\llbracket c_6, c_{13} \rrbracket = b_4 - \frac{1}{2}b_{11}$	$\llbracket c_8, c_{12} \rrbracket = b_5$
$\llbracket c_2, c_{13} \rrbracket = b_8 + \frac{1}{2}b_9$	$\llbracket c_4, c_{15} \rrbracket = -b_5$	$\llbracket c_6, c_{14} \rrbracket = -b_{15}$	$\llbracket c_8, c_{13} \rrbracket = -b_8 - \frac{1}{2}b_{12}$
$\llbracket c_2, c_{14} \rrbracket = -b_7 + b_{13}$	$\llbracket c_4, c_{16} \rrbracket = -b_7 + b_{16}$	$\llbracket c_7, c_{10} \rrbracket = \frac{1}{2}b_{10}$	$\llbracket c_8, c_{14} \rrbracket = b_7 - b_{16}$

TABLE 5. Nijenhuis–Richardson  $\bullet : C^2 \times C^2 \rightarrow C^3$ 

$\bullet$	$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$	$c_{11}$	$c_{12}$	$c_{13}$	$c_{14}$	$c_{15}$	$c_{16}$
$c_1$	0	0	0	0	0	0	0	0	$b_1$	$b_2$	$b_5$	$b_6$	$b_7$	$b_8$	0	0
$c_2$	0	0	0	0	0	0	0	0	0	0	$b_6$	$-b_5$	$b_8$	$-b_7$	0	0
$c_3$	0	0	0	0	0	0	0	0	0	0	0	$b_1$	0	$b_2$	$b_6$	$b_8$
$c_4$	0	0	0	0	0	0	0	0	0	0	$-b_1$	0	$-b_2$	0	$-b_5$	$-b_7$
$c_5$	0	0	0	0	0	0	0	0	$b_6$	$b_8$	0	$b_3$	0	$b_4$	0	0
$c_6$	0	0	0	0	0	0	0	0	$b_5$	$b_7$	$b_3$	0	$b_4$	0	0	0
$c_7$	0	0	0	0	0	0	0	0	0	0	$b_5$	$b_6$	$b_7$	$b_8$	$b_3$	$b_4$
$c_8$	0	0	0	0	0	0	0	0	0	0	$-b_6$	$b_5$	$-b_8$	$b_7$	0	0
$c_9$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_{10}$	0	0	0	0	$\frac{1}{2}b_9$	$-b_{13}$	$\frac{1}{2}b_{10}$	$-b_{14}$	0	0	0	0	0	0	0	0
$c_{11}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_{12}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_{13}$	$b_{13}$	$\frac{1}{2}b_9$	$b_{14}$	$\frac{1}{2}b_{10}$	$b_{15}$	$-\frac{1}{2}b_{11}$	$b_{16}$	$-\frac{1}{2}b_{12}$	0	0	0	0	0	0	0	0
$c_{14}$	$-\frac{1}{2}b_9$	$b_{13}$	$-\frac{1}{2}b_{10}$	$b_{14}$	$-\frac{1}{2}b_{11}$	$-b_{15}$	$-\frac{1}{2}b_{12}$	$-b_{16}$	0	0	0	0	0	0	0	0
$c_{15}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_{16}$	$\frac{1}{2}b_{11}$	$b_{15}$	$\frac{1}{2}b_{12}$	$b_{16}$	0	0	0	0	0	0	0	0	0	0	0	0

**A.2. Enumeration of the complex deformation complex.** We shall now enumerate bases for  $C_C^p$ ,  $p = 0, 1, 2, 3$ .  $C_C^0$  is spanned by  $R$  and  $H$ . A basis for  $C_C^1$  is given by  $a_1, a_2, a_3, \bar{a}_3, a_4, \bar{a}_4, a_5, \bar{a}_5, a_6$  and  $\bar{a}_6$ , where the  $a_i$  are defined in Table 6. A basis for  $C_C^2$  is given by  $c_1, \bar{c}_1, c_2, \bar{c}_2, c_3, \bar{c}_3, c_4, \bar{c}_4, c_5, c_6, c_7, \bar{c}_7, c_8, \bar{c}_8, c_9$  and  $c_{10}$ , where the  $c_i$  are defined in Table 7. Finally, as basis for  $C_C^3$  is given by  $b_1, b_2, b_3, \bar{b}_3, b_4, \bar{b}_4, b_5, b_6, b_7, \bar{b}_7, b_8, \bar{b}_8, b_9, \bar{b}_9, b_{10}$  and  $\bar{b}_{10}$ , where the  $b_i$  are defined in Table 8. The complex conjugates of the basis elements are the naive ones, e.g.,  $\bar{c}_1 = \eta\bar{\beta}\bar{B}$ .

TABLE 6. Basis for  $C^1(\mathfrak{h}_C; \mathfrak{g}_C)^\tau$ 

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$
$\eta R$	$\eta H$	$\beta B$	$\beta P$	$\pi B$	$\pi P$

TABLE 7. Basis for  $C^2(\mathfrak{h}_C; \mathfrak{g}_C)^\tau$ 

$c_1$	$c_2$	$c_3$	$c_4$	$c_5$	$c_6$	$c_7$	$c_8$	$c_9$	$c_{10}$
$\eta\beta B$	$\eta\beta P$	$\eta\pi B$	$\eta\pi P$	$i\beta\bar{\beta}R$	$i\beta\bar{\beta}H$	$\beta\bar{\pi}R$	$\beta\bar{\pi}H$	$i\pi\bar{\pi}R$	$i\pi\bar{\pi}H$

TABLE 8. Basis for  $C^3(\mathfrak{h}_C; \mathfrak{g}_C)^\tau$ 

$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$b_6$	$b_7$	$b_8$	$b_9$	$b_{10}$
$i\eta\beta\bar{\beta}R$	$i\eta\beta\bar{\beta}H$	$\eta\beta\bar{\pi}R$	$\eta\beta\bar{\pi}H$	$i\eta\pi\bar{\pi}R$	$i\eta\pi\bar{\pi}H$	$i\beta\bar{\beta}\pi B$	$i\beta\bar{\beta}\pi P$	$i\beta\pi\bar{\pi}B$	$i\beta\pi\bar{\pi}P$

Finally we work out the Nijenhuis–Richardson bracket  $C_C^2 \times C_C^2 \rightarrow C_C^3$ . Table 9 displays the multiplication table for  $\bullet : C_C^2 \times C_C^2 \rightarrow C_C^3$  from where we obtain the Nijenhuis–Richardson bracket by symmetrisation.

TABLE 9. Nijenhuis–Richardson  $\bullet : C_{\mathbb{C}}^2 \times C_{\mathbb{C}}^2 \rightarrow C_{\mathbb{C}}^3$ 

$\bullet$	$c_1$	$\bar{c}_1$	$c_2$	$\bar{c}_2$	$c_3$	$\bar{c}_3$	$c_4$	$\bar{c}_4$	$c_5$	$c_6$	$c_7$	$\bar{c}_7$	$c_8$	$\bar{c}_8$	$c_9$	$c_{10}$
$c_1$	0	0	0	0	0	0	0	0	$b_1$	$b_2$	$b_3$	0	$b_4$	0	0	0
$\bar{c}_1$	0	0	0	0	0	0	0	0	$b_1$	$b_2$	0	$\bar{b}_3$	0	$\bar{b}_4$	0	0
$c_2$	0	0	0	0	0	0	0	0	0	0	0	$ib_1$	0	$ib_2$	$ib_3$	$ib_4$
$\bar{c}_2$	0	0	0	0	0	0	0	0	0	0	$-ib_1$	0	$-ib_2$	0	$-i\bar{b}_3$	$-i\bar{b}_4$
$c_3$	0	0	0	0	0	0	0	0	$-i\bar{b}_3$	$-i\bar{b}_4$	$-ib_5$	0	$-ib_6$	0	0	0
$\bar{c}_3$	0	0	0	0	0	0	0	0	$ib_3$	$ib_4$	0	$ib_5$	0	$ib_6$	0	0
$c_4$	0	0	0	0	0	0	0	0	0	0	0	$\bar{b}_3$	0	$\bar{b}_4$	$b_5$	$b_6$
$\bar{c}_4$	0	0	0	0	0	0	0	0	0	0	$b_3$	0	$b_4$	0	$b_5$	$b_6$
$c_5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_6$	0	0	0	0	$b_7$	$\bar{b}_7$	$b_8$	$\bar{b}_8$	0	0	0	0	0	0	0	0
$c_7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\bar{c}_7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_8$	0	$i\bar{b}_7$	0	$i\bar{b}_8$	$ib_9$	0	$ib_{10}$	0	0	0	0	0	0	0	0	0
$\bar{c}_8$	$-ib_7$	0	$-ib_8$	0	0	$-i\bar{b}_9$	0	$-i\bar{b}_{10}$	0	0	0	0	0	0	0	0
$c_9$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$c_{10}$	$b_9$	$\bar{b}_9$	$b_{10}$	$\bar{b}_{10}$	0	0	0	0	0	0	0	0	0	0	0	0

The nonzero Nijenhuis–Richardson brackets are the following:

$$\begin{aligned}
\llbracket c_1, c_5 \rrbracket &= b_1 & \llbracket c_2, \bar{c}_7 \rrbracket &= ib_1 & \llbracket c_3, c_5 \rrbracket &= -i\bar{b}_3 & \llbracket c_4, c_6 \rrbracket &= b_8 \\
\llbracket c_1, c_6 \rrbracket &= b_2 & \llbracket c_2, \bar{c}_8 \rrbracket &= i(b_2 - b_8) & \llbracket c_3, c_6 \rrbracket &= -i\bar{b}_4 + b_7 & \llbracket c_4, c_7 \rrbracket &= \bar{b}_3 \\
\llbracket c_1, c_7 \rrbracket &= b_3 & \llbracket c_2, c_9 \rrbracket &= ib_3 & \llbracket c_3, c_7 \rrbracket &= -ib_5 & \llbracket c_4, c_8 \rrbracket &= ib_{10} \\
\llbracket c_1, c_8 \rrbracket &= b_4 & \llbracket c_2, c_{10} \rrbracket &= ib_4 + b_{10} & \llbracket c_3, c_8 \rrbracket &= -i(b_6 - b_9) & \llbracket c_4, \bar{c}_8 \rrbracket &= \bar{b}_4 \\
\llbracket c_1, \bar{c}_8 \rrbracket &= -ib_7 & & & & & \llbracket c_4, \bar{c}_7 \rrbracket &= b_5 \\
& & & & & & \llbracket c_4, \bar{c}_7 \rrbracket &= b_6
\end{aligned} \tag{114}$$

and their complex conjugates, which we do not list explicitly. For example,  $\llbracket \bar{c}_1, c_5 \rrbracket = b_1$ , et cetera, using that  $\llbracket \bar{\lambda}, \bar{\mu} \rrbracket = \overline{\llbracket \lambda, \mu \rrbracket}$ .

**A.3. Dictionary between the two enumerations.** For ease of translation between the complex and real enumerations, we provide the following dictionary for the first two spaces of cochains. For  $C^1$  we have

$$\begin{aligned}
a_1 &= a_1 & a_1 &= a_1 \\
a_2 &= a_2 & a_2 &= a_2 \\
a_3 &= a_3 + \bar{a}_3 & a_3 &= \frac{1}{2}(a_3 + ia_4) \\
a_4 &= -i(a_3 - \bar{a}_3) & a_4 &= \frac{1}{2}(a_5 + ia_6) \\
a_5 &= a_4 + \bar{a}_4 & a_5 &= \frac{1}{2}(a_7 + ia_8) \\
a_6 &= -i(a_4 - \bar{a}_4) & a_6 &= \frac{1}{2}(a_9 + ia_{10}) \\
a_7 &= a_5 + \bar{a}_5 & & \\
a_8 &= -i(a_5 - \bar{a}_5) & & \\
a_9 &= a_6 + \bar{a}_6 & & \\
a_{10} &= -i(a_6 - \bar{a}_6) & &
\end{aligned} \tag{115}$$

and for  $C^2$  we have

$$\begin{aligned}
c_1 &= c_1 + \bar{c}_1 \\
c_2 &= -i(c_1 - \bar{c}_1) \\
c_3 &= c_2 + \bar{c}_2 \\
c_4 &= -i(c_2 - \bar{c}_2) \\
c_5 &= c_3 + \bar{c}_3 \\
c_6 &= -i(c_3 - \bar{c}_3) \\
c_7 &= c_4 + \bar{c}_4 \\
c_8 &= -i(c_4 - \bar{c}_4) \\
c_9 &= -2c_5 \\
c_{10} &= -2c_6 \\
c_{11} &= 2(c_7 + \bar{c}_7) \\
c_{12} &= -2i(c_7 - \bar{c}_7) \\
c_{13} &= 2(c_8 + \bar{c}_8) \\
c_{14} &= -2i(c_8 - \bar{c}_8) \\
c_{15} &= -2c_9 \\
c_{16} &= -2c_{10} \\
c_1 &= \frac{1}{2}(c_1 + ic_2) \\
c_2 &= \frac{1}{2}(c_3 + ic_4) \\
c_3 &= \frac{1}{2}(c_5 + ic_6) \\
c_4 &= \frac{1}{2}(c_7 + ic_8) \\
c_5 &= -\frac{1}{2}c_9 \\
c_6 &= -\frac{1}{2}c_{10} \\
c_7 &= \frac{1}{4}(c_{11} + ic_{12}) \\
c_8 &= \frac{1}{4}(c_{13} + ic_{14}) \\
c_9 &= -\frac{1}{2}c_{15} \\
c_{10} &= -\frac{1}{2}c_{16}.
\end{aligned} \tag{116}$$

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